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# Switching Diffusion Systems With Past-Dependent Switching Having A Countable State Space

Hai Dang Nguyen
*Wayne State University,*

# SWITCHING DIFFUSION SYSTEMS WITH PAST-DEPENDENT SWITCHING HAVING A COUNTABLE STATE SPACE

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**HAI DANG NGUYEN**

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# DEDICATION

To my uncle, Pham Van Viet.

In memory of my grandmother, who passed away when I just started my PhD program over 8000 miles away.

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# CHAPTER 1 INTRODUCTION

## 1.1 Introduction

Owing to the demand of modeling, analysis, and computation of complex networked systems, much attention has been devoted to building more realistic dynamic system models. It has been well recognized that in many real-world applications, traditional models using continuous processes represented by solutions to deterministic differential equations and stochastic differential equations alone are often inadequate. Arising from control engineering, queueing networks, manufacturing and production planning, parameter estimation, filtering of dynamic systems, ecological and biological systems, and financial engineering, etc., numerous complex systems contain both continuous dynamics and discrete events. The discrete events in these systems are not normally representable by solutions of the usual differential equations. Because of the demand, switching diffusions (also known as hybrid switching diffusions) have drawn growing and resurgent attention. A switching diffusion is a two-component process $(X(t), \alpha(t))$ in which the continuous component $X(t)$ evolves according to the diffusion process whose drift and diffusion coefficients depend on the state of $\alpha(t)$, whereas $\alpha(t)$ takes values in a set consisting of isolated points. Because of their importance, many papers have been devoted to such hybrid dynamic systems; see [26, 45, 61, 57] and the references therein. In their comprehensive treatment of hybrid switching diffusions, Mao and Yuan [31] focused on $\alpha(t)$ being a continuous-time and homogeneous Markov chain independent of the Brownian motion and the generator of the Markov chain being a constant matrix. Realizing the need, treating the two components jointly, Yin and Zhu [60] extended the study to the Markov process $(X(t), \alpha(t))$ by allowing the generator $\alpha(t)$ to depend on the current state $X(t)$. Properties of the underlying process including recurrence, positive recurrence, ergodicity, Feller properties, stability, and invariance among others were investigated. Such study provides us with a clear picture of the underlying processes. Nevertheless, in both of the aforementioned books and most related papers to date, the switching process $\alpha(t)$ is assumed to have a finite state space. One question naturally arises. What happens if the switching process has a countable state space? Much of the argument in [60] relies on the interplay of stochastic processes and the associated systems of partial differential equations. Because the state space of $\alpha(t)$ was assumed to be a finite set, one can essentially treat a system of partial differential equations with a finite number of equations. When we consider problems involving a countable state space, the number of equations becomes infinite. Much more complex situation is encountered. Different methods have to be developed to treat the systems.

There are plenty of real-world applications involving such switching diffusions. Perhaps, one of the most widely used control models in the literature is the so-called LQG (linear quadratic Gaussian regulator) problem; see [14, pp.165-166] for a traditional model. However, for many new applications in networked systems, it has been found that in addition to the random noise represented by Brownian type of disturbances, there is a source of randomness owing to the presence of random environment that can be modeled by a continuous-time Markov chain. Let $\alpha(t)$ be a continuous-time Markov chain with state space $\mathbb{Z}_+$ (the set of positive integers) and generator $Q$. Consider the controlled dynamic system

$$\begin{aligned} &dX(t) = [A(\alpha(t))X(t) + B(\alpha(t))u(t)]dt + \sigma(\alpha(t))dW(t), \\ &X(s) = x, \text{ for } s \le t \le T, \end{aligned} \tag{1.1}$$

where $X(t) \in \mathbb{R}^{n_1}$ is the continuous state variable, $u(t) \in \mathbb{R}^{n_2}$ is the control, $A(i) \in \mathbb{R}^{n_1 \times n_1}$ and $B(i) \in \mathbb{R}^{n_1 \times n_2}$ are well defined and have finite values for each $i \in \mathbb{Z}_+$. One may wish to find the optimal control $u(\cdot)$ so that the expected quadratic cost function

$$J(s,i,x,u(\cdot)) = E\Big[\int_s^T [X^\top(t)M(\alpha(t))X(t) + u^\top(t)N(\alpha(t))u(t)]dt + X^\top(T)DX(T)\Big] \tag{1.2}$$

is minimized. The use of $\alpha(t)$ stems from the formulation of discrete events, and the use of $\mathbb{Z}_+$ enlarges the applicability of previous consideration of finite state space cases. Switched dynamic systems can also be found in, for example, modeling impatient customers and customer abandonment of Markov-modulated service speeds in the heavy-traffic regime and the many-server systems in the Halfin-Whitt regime and the non-degenerate slowdown regime; see [18]. We also refer the reader to Whitt [52] for further reading on limit results in queueing theory and many references therein. In fact, in most of the queueing models, the discrete set is countable rather than finite.

In another example, we consider an extension of the Markov-modulated-rate fluid models treated in [63]. Stemming from queueing systems, this example is simple in that it is even without the Brownian motion part, but it explains the modeling view point of the past depend switching with a countable state space. Consider the fluid buffer model with an infinite capacity. Let $X(t)$ be the amount of fluids in the buffer at time $t$, known as buffer content or buffer level. The fluids enter and leave the buffer at random rates. The input and output of fluids are modulated by a switching process $\alpha(t)$ with state space $\mathbb{Z}_+ = \{1, 2, \ldots\}$, known as a stochastic external environment. Using $\alpha(t)$ to determine the input and output rates, we introduce a drift function $f(\cdot)$ (see Kulkarni [28]): $f(\cdot) : \mathbb{Z}_+ \mapsto (-\infty, \infty)$. Different from [63], we do not assume $\alpha(t)$ to be a Markov chain, but rather, we assume that the transition rates satisfy (1.6). That is, the transition rates depends on the history of the process $X(t)$.

Note that formally, the net rate that is the difference of the input and the output rates at time $t$ is given by $f(\alpha(t))$. The dynamics of the buffer content $\{X(t) : t \geq 0\}$ can be described by the following differential equation:

$$\begin{aligned} \frac{d}{dt}X(t) &= \begin{cases} f(\alpha(t)), & \text{if } X(t) > 0 \\ (f(\alpha(t)))^+, & \text{if } X(t) = 0, \end{cases} \\ &= f(\alpha(t))\mathbf{1}_{\{X(t)>0\}} + (f(\alpha(t)))^+\mathbf{1}_{\{X(t)=0\}}, \end{aligned} \tag{1.3}$$

where $x^+ = \max\{x, 0\}$. Note that $X(t)$ can be rewritten as

$$X(t) = X(0) + Y(t) - \left(\inf_{0 \leq s \leq t} \{Y(s) + X(0)\}\right) \wedge 0 \tag{1.4}$$

with

$$Y(s) = \int_0^s f(\alpha(v))dv,$$

where $a \wedge b = \min(a, b)$ for two real numbers $a$ and $b$, and

$$-(\inf_{0 \leq s \leq t} \{Y(s) + X(0)\}) \wedge 0$$

measures the amount of potential output lost up to time $t$ due to the emptiness of the buffer.

Many of the current interests are concerned with the fluid model above such as long-run average control problems or stability of the systems.

Two more dynamic systems are to be presented in the next section, in which the main interests are to find long-term behavior and control design in an ecological system and to find optimal strategies under long-run average criteria for a pollution management problem. In order to study the aforesaid problems, we first need to ensure that the systems under consideration have unique solutions and that the solutions possess good properties. Motivated by these examples, we take up the challenge of considering a nonlinear hybrid diffusion $(X(t), \alpha(t))$ whose discrete component $\alpha(t)$ has an infinite state space in this paper. Moreover, in lieu of allowing the switching process to depend on the current state $X(t)$ only, we assume that it is past dependent. That is, we assume that the generator of $\alpha(t)$ depends on the past history of the continuous process. This paper provides conditions for the existence and uniqueness of the solutions for given initial data, and to demonstrate the Markov-Feller property of a function-valued stochastic process associated with the equation. Our study will build a bridge for future study on related control systems.

The rest of this dissertation is organized as follows. The formulation and some more examples of switching diffusions with past-dependent switching and countably many possible switching locations is given in Sections 1.2 and 1.3 of Chapter 1. The existence and uniqueness of solutions to the stochastic equations are then proved under suitable conditions in Section 3.1 of Chapter 3. Section 3.2 studies the Markov and Feller properties of a function-valued stochastic process associated with our equation. The proof for the Feller property is rather complex because the state space of $\alpha(t)$ is infinite, the space of continuous functions is not locally compact, and we do not assume uniform continuity of the switching intensities. In Chapter 4, we consider the recurrence and ergodicity of the process. Chapter 5 provides conditions for stability and instability of the system.

## 1.2 Formulation

Let $r$ be a fixed positive number. Denote by $\mathcal{C}([a,b],\mathbb{R}^n)$ the set of $\mathbb{R}^n$-valued continuous functions defined on $[a,b]$. In what follows, we mainly work with $\mathcal{C}([-r,0],\mathbb{R}^n)$, and simply denote it by $\mathcal{C}:=\mathcal{C}([-r,0],\mathbb{R}^n)$. Denote by $|x|$ the Euclidean norm of $x\in\mathbb{R}^n$. For $\phi\in\mathcal{C}$, we use the norm $\|\phi\|=\sup\{|\phi(t)|:t\in[-r,0]\}$. For $y(\cdot)\in\mathcal{C}([-r,\infty),\mathbb{R}^n)$ and $t\geq 0$, we denote by $y_t$ the so-called segment function (or memory segment function) $y_t(\cdot):=y(t+\cdot)\in\mathcal{C}$. Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ be a complete filtered probability space with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual condition, i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $W(t)$ be an $\mathcal{F}_t$-adapted and $\mathbb{R}^d$-valued Brownian motion. Suppose $b(\cdot,\cdot):\mathbb{R}^n\times\mathbb{Z}_+\to\mathbb{R}^n$ and $\sigma(\cdot,\cdot):\mathbb{R}^n\times\mathbb{Z}_+\to\mathbb{R}^{n\times d}$, where $\mathbb{Z}_+=\mathbb{N}\setminus\{0\}=\{1,2,\dots\}$, the set of positive integers. Consider the two-component process $(X(t),\alpha(t))$, where $\alpha(t)$ is a pure jump process taking value in $\mathbb{Z}_+$, and $X(t)$ satisfies

$$dX(t)=b(X(t),\alpha(t))dt+\sigma(X(t),\alpha(t))dW(t). \tag{1.5}$$

We assume that if $\alpha(t-):=\lim_{s\to t^-}\alpha(s)=i$, then it can switch to $j$ at $t$ with intensity $q_{ij}(X_t)$ where $q_{ij}(\cdot):\mathcal{C}\to\mathbb{R}$. When $q_i(\phi):=\sum_{j=1,j\neq i}^{\infty}q_{ij}(\phi)$ is uniformly bounded in $(\phi,i)\in\mathcal{C}\times\mathbb{Z}_+$, and $q_i(\cdot)$ and $q_{ij}(\cdot)$ are continuous, one may view the aforementioned assumption as

$$\begin{aligned} &\mathbb{P}\{\alpha(t+\Delta)=j|\alpha(t)=i, X_s, \alpha(s), s\le t\} = q_{ij}(X_t)\Delta + o(\Delta) \text{ if } i\ne j \text{ and} \\ &\mathbb{P}\{\alpha(t+\Delta)=i|\alpha(t)=i, X_s, \alpha(s), s\le t\} = 1 - q_i(X_t)\Delta + o(\Delta). \end{aligned} \tag{1.6}$$

However, when $q_i(\phi)$ and $q_{ij}(\phi)$ are either discontinuous or unbounded, it does not seem appropriate to use (1.6) to model the switching intensity. To formulate the problem in a general setting without the boundedness and continuity assumptions mentioned above, we construct $\alpha(t)$ as the solution to a stochastic differential equation with respect to a Poisson random measure. We elaborate on the idea below. Let $\mathfrak{p}(dt, dz)$ be a Poisson random measure with intensity $dt\times\mathfrak{m}(dz)$ and $\mathfrak{m}$ is the Lebesgue measure on $\mathbb{R}$ such that $\mathfrak{p}(\cdot,\cdot)$ is independent of the Brownian motion $W(\cdot)$. Let $\tilde{\mathfrak{p}}$ be the Poisson point process associated with $\mathfrak{p}(\cdot,\cdot)$ (see e.g., [48]). Then $\tilde{\mathfrak{p}}$ can lie in a set $A$ with intensity $\mathfrak{m}(A)$, that is, the expected number of Poisson points lying in $A$ during the period $dt$ is $dt\times\mathfrak{m}(A)$. Using this fact, for each $i\in\mathbb{Z}$, we can construct disjoint sets $\{\Delta_{ij}(\phi), j\ne i\}$ such that $\mathfrak{m}(\Delta_{ij}(\phi)) = q_{ij}(\phi)$. Let $\tilde{\mathfrak{p}}$ govern the switching of $\alpha(t)$ in the manner that if $\alpha(t-)=i$ and there is a Poisson point in $\Delta_{ij}(X_t)$ at time $t$, then $\alpha(t)=j$. If $\alpha(t-)=i$ and there is no Poisson point in $\cup_{j\ne i}\Delta_{ij}(X_t)$ at time $t$, $\alpha(t)$ remains $i$. Using this idea, we formulate the equation for $\alpha(t)$ as follows. For each function $\phi:[-r,0]\to\mathbb{R}^n$, and $i\in\mathbb{Z}_+$, let $\Delta_{ij}(\phi), j\ne i$ be the consecutive left-closed and right-open intervals of the real line, each having length $q_{ij}(\phi)$. That is,

$$\Delta_{i1}(\phi) = [0, q_{i1}(\phi)),\ \Delta_{ij}(\phi) = \Big[\sum_{k=1,k\ne i}^{j-1} q_{ik}(\phi), \sum_{k=1,k\ne i}^{j} q_{ik}(\phi)\Big), j>1, j\ne i.$$

Define $h:\mathcal{C}\times\mathbb{Z}_+\times\mathbb{R}\mapsto\mathbb{R}$ by $h(\phi,i,z)=\sum_{j=1,j\ne i}^{\infty}(j-i)\mathbf{1}_{\{z\in\Delta_{ij}(\phi)\}}$, where $\mathbf{1}_{\{z\in\Delta_{ij}(\phi)\}}=1$ if $z\in\Delta_{ij}$, otherwise $\mathbf{1}_{\{z\in\Delta_{ij}(\phi)\}}=0$, is the indicator function. The process $\alpha(t)$ can be defined as a solution to

$$d\alpha(t) = \int_{\mathbb{R}} h(X_t, \alpha(t-), z)\mathfrak{p}(dt, dz).$$

The pair $(X(t), \alpha(t))$ is therefore a solution to the system of equations

$$\begin{cases} dX(t) = b(X(t), \alpha(t))dt + \sigma(X(t), \alpha(t))dW(t) \\ d\alpha(t) = \int_{\mathbb{R}} h(X_t, \alpha(t-), z)\mathfrak{p}(dt, dz). \end{cases} \tag{1.7}$$

A strong solution to (1.7) on $[0, T]$ with initial data $(\phi, i_0)$ being $\mathcal{C} \times \mathbb{Z}_+$-valued and $\mathcal{F}_0$-measurable random variable, is an $\mathcal{F}_t$-adapted process $(X(t), \alpha(t))$ such that

- $X(t)$ is continuous and $\alpha(t)$ is cadlag (right continuous with left limits) almost surely (a.s.).
- $X(t) = \phi(t)$ for $t \in [-r, 0]$ and $\alpha(0) = i_0$
- $(X(t), \alpha(t))$ satisfies (1.7) for all $t \in [0, T]$ a.s.

We will show in the Appendix that the solution $(X(t), \alpha(t))$ to (1.7), satisfies (1.6) under suitable conditions. Let $f(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{Z}_+ \mapsto \mathbb{R}$ be twice continuously differentiable in $x$ and bounded in $(x, i) \in \mathbb{R}^n \times \mathbb{Z}_+$. We define the "operator" $\mathcal{L}f(\cdot, \cdot) : \mathcal{C} \times \mathbb{Z}_+ \mapsto \mathbb{R}$ by

$$\begin{aligned} \mathcal{L}f(\phi, i) =& \nabla f(\phi(0), i)b(\phi(0), i) + \frac{1}{2}\operatorname{tr}\left(\nabla^2 f(\phi(0), i)A(\phi(0), i)\right) \\ &+ \sum_{j=1, j\neq i}^{\infty} q_{ij}(\phi)\big[f(\phi(0), j) - f(\phi(0), i)\big] \\ =& \sum_{k=1}^{n} b_k(\phi(0), i)f_k(\phi(0), i) + \frac{1}{2}\sum_{k,l=1}^{n} a_{kl}(\phi(0), i)f_{kl}(\phi(0), i) \\ &+ \sum_{j=1, j\neq i}^{\infty} q_{ij}(\phi)\big[f(\phi(0), j) - f(\phi(0), i)\big], \end{aligned} \tag{1.8}$$

where $b(x, i) = (b_1(x, i), \dots, b_n(x, i))^\top$, $\nabla f(x, i) = (f_1(x, i), \dots, f_n(x, i)) \in \mathbb{R}^{1\times n}$ and

$\nabla^2 f(x,i) = (f_{ij}(x,i))_{n\times n}$ are the gradient and Hessian of $f(x,i)$ with respect to $x$, respectively, with

$$f_k(x,i) = (\partial/\partial x_k)f(x,i),\ f_{kl}(x,i) = (\partial^2/\partial x_k\partial x_l)f(x,i), \quad \text{and}$$
$$A(x,i) = (a_{kl}(x,i))_{n\times n} = \sigma(x,i)\sigma^\top(x,i),$$

with $z^\top$ denoting the transpose of $z$. We have Itô's formula:

$$f(X(t),\alpha(t)) - f(X(0),\alpha(0)) = \int_0^t \mathcal{L}f(X_s,\alpha(s-))ds + M_1(t) + M_2(t) \ \text{a.s.}, \tag{1.9}$$

where $M_1(\cdot)$ and $M_2(\cdot)$ are local martingales, defined by

$$\begin{aligned} M_1(t) &= \int_0^t \nabla f(X(s),\alpha(s-))\sigma(X(s),\alpha(s-))dW(s),\\ M_2(t) &= \int_0^t\int_{\mathbb{R}} \big[f\big(X(s),\alpha(s-)+h(X_s,\alpha(s-),z)\big) - f(X(s),\alpha(s-))\big]\mu(ds,dz). \end{aligned} \tag{1.10}$$

It should be noted that $\mathcal{L}$ is not the generator of the Markov process $(X_t,\alpha(t))$. However this operator is very useful for analyzing the process $(X(t),\alpha(t))$. In view of (1.10), if $\tau_1 \le \tau_2$ are stopping times that are bounded above by $T$ a.s., and $f(\cdot,\cdot)$ and $\mathcal{L}f(\cdot,\cdot)$ are bounded and $q_{\alpha(t)}(X_t) < \infty$ on $[\tau_1,\tau_2]$, then

$$\mathbb{E}f(X(\tau_2),\alpha(\tau_2)) = \mathbb{E}f(X(\tau_1),\alpha(\tau_1)) + \mathbb{E}\int_{\tau_1}^{\tau_2} \mathcal{L}f(X_t,\alpha(t-))dt.$$

**Remark 1.1.** If $\alpha(t)$ depends on the continuous state, but there is no past dependence (that is, $X_t$ is replaced by $X(t)$ in (1.6), and $\phi$ and $\phi(0)$ are replaced by the current state $X(t) = x$ in (1.8), respectively), then $\mathcal{L}$ is indeed the generator of the process $(X(t),\alpha(t))$. Even in this case, the current paper settles the matter of the state space of the switching process being countable thus generalizes the study of finite state space cases as considered in [60].

## 1.3 Examples

**Example 1.2.** This example stems from applications in ecological systems and biological control. Consider the evolution of two interacting species. One is micro, which is described by a logistic differential equation perturbed by a white noise. The other is macro, we assume that its number of individuals follows a birth-death process. Let $X(t)$ be the density of the micro species and $\alpha(t)$ the population of the macro species. The life cycle of a micro species is usually very short, so it is reasonable to assume that the evolution of $X(t)$ can be described by the following past-independent equation

$$dX(t) = X(t)\big[a(\alpha(t)) - b(\alpha(t))X(t)\big]dt + \sigma(\alpha(t))X(t)dW(t), \tag{1.11}$$

where $a(i), b(i), \sigma(i)$ are positive constants for each $i \in \mathbb{Z}_+$.

On the other hand, the reproduction process of $\alpha(t)$ is assumed to be non-instantaneous. More precisely, suppose the reproduction depends on the period of time from egg formation to hatching, say $r$. Then we have

$$d\alpha(t) = \int_{\mathbb{R}} h(X_t, \alpha(t-), z)\mathfrak{p}(dt, dz), \tag{1.12}$$

where $h(\phi, i, z) = \sum_{j=1, j\neq i}^{\infty}(j - i)\mathbf{1}_{\{z\in\Delta_{ij}(\phi)\}}$, $\Delta_{i,i+1}(\phi) = [0, \beta_i(\phi))$, $\Delta_{i,i-1}(\phi) = [0, \delta_i(\phi))$, $\Delta_{i,j}(\phi) = \emptyset$ if $j \notin \{i-1, i, i+1\}$ or $i = 0$. Usually $\beta_i(\phi), \delta_i(\phi)$ can be given in the integral from $\beta_i(\phi) = \int_{-r}^{0} \tilde{\beta}_i(t)\phi(t)dt$, $\delta_i(\phi) = \int_{-r}^{0} \tilde{\delta}_i(t)\phi(t)dt$, for some appropriate weighting functions $\tilde{\beta}_i, \tilde{\delta}_i$. As can be seen from the above, the switching process at $t$ in fact depends on past history of the state $X(\cdot)$. Investigating the interactions between the two species are very important to biological control. A basic biological control problem aims to choose a suitable living organism to control a particular pest (see e.g., [19, 30]). This chosen organism might be a predator, parasite, or disease, which will attack the harmful insect. To design and evaluate effectiveness of a biological control, some questions should be answered first. For example, under which conditions the species will be permanent forever or they will extinct at some instance? Whether or not there is an invariant measure associated with the system under consideration. Mathematically, these questions are related to the stability and ergodicity of the corresponding stochastic systems, which will be studied in a future paper.

**Example 1.3.** Pollution management is vitally important and has a significant impact on environment. A major issue is concerned with the tradeoff of pollution accumulation and consumption, which affects environmental policy making. Following the seminal paper of Keeler et al. [23], much work has been devoted to the study of optimal control of dynamic economic systems. In [22], Kawaguchi and Morimoto treated a pollution accumulation problem of maximizing the long-run average welfare using a controlled diffusion model. Assume that an economy consumes some good and meanwhile generates pollution. The pollution stock is gradually degraded and its instantaneous growth rate incorporates a random disturbance with mean zero and constant variance. The social welfare is defined by the utility of the consumption net of the disutility of pollution. The problem is to find optimal consumption strategies for the society in the long-run average sense. Departing from their formulation, we consider an extension of their model. Suppose that there is a switching process $\alpha(t)$ taking values in $\mathbb{Z}_+$ such that $\alpha(t)$ represents the level of pollution at time $t$. Assume that the stock of pollution at time $t$ is given by $X(t)$, a real-valued process, and there is a positive real-valued function $\rho(\cdot)$ so that for each $i \in \mathbb{Z}_+$, the rate of pollution decay is $\rho(i)$. The consumption rate (or flow of pollution) is a control process, which is denoted by $c(t)$ at time $t$; the social utility function of the consumption $c$ is denoted by $U(c)$, whereas the social disutility of the pollution stock $x$ is $D(x)$. We say that the consumption rate is admissible if it is $\mathcal{F}_t$-measurable, where $\mathcal{F}_t = \{(X(s), \alpha(s)) : s \le t\}$ such that $0 \le c(t) \le K_0$ for some $K_0 > 0$. The ultimate objective is to maximize the long-run average welfare

$$J(c(\cdot)) = \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T [U(c(t)) - D(X(t))]dt, \tag{1.13}$$

subject to

$$dX(t) = [c(t) - \rho(\alpha(t))X(t)]dt + \sigma(X(t), \alpha(t))dW(t). \tag{1.14}$$

Assume that the pollution level $\alpha(\cdot)$ satisfies the conditions (1.6). First, it is reasonable that the level of pollution can be modeled by a continuous-time process taking values in $\mathbb{Z}_+$. Second, to be more realistic, the pollution level depends on the pollution stock $X(t)$ as well as some past history as given in (1.6). As another generalization of [22], we assume that $\sigma$ in fact depends on $(X(t), \alpha(t))$, and the switching rate depends on some past history of the pollution stock $X(\cdot)$ as in (1.6), and $\sigma^2(x, i) > 0$ for each $i \in \mathbb{Z}_+$. Treating the optimal pollution management problem, it is natural to consider the replacement of the average in (1.13) by the average with respect to an invariant measure (if it exists) of the controlled systems. To do so, we need to make sure that (1.14) indeed has an invariant measure. Before this matter can be settled, we need to show that the system has a unique solution for each initial data, and the solution possesses certain desired properties such as Markov and Feller properties.

# CHAPTER 2 WELL-POSEDNESS AND MARKOV-FELLER PROPERTIES OF SOLUTIONS

## 2.1 Existence and Uniqueness of Solutions

We are now in a position to prove the existence and uniqueness in the strong sense of a solution with given initial data under suitable conditions. We give several sets of conditions. The main reason is due to the past dependence and the use of $\mathbb{Z}_+$. First in contrast to the case of switching process staying in a finite set, care needs to be exercised regarding uniformity with respect to the switching set. Second, the past dependence requires careful handling of the use of Lipschitz continuity etc. and the uniformity with respect to the element in the corresponding function spaces. Depending on the preference, Assumptions 2.1 allows certain bounds to be dependent of the switching state $i$, but uniform in the variable in the function space, whereas Assumption 2.2 requires uniformity in the bounds w.r.t. $i$, but requires the past dependent part be localized. Assumptions 2.3 and 2.4 relax the Lipschitz condition to local Lipschitz together with certain growth conditions presented by using bounds with the help of Lyapunov functions.

**Assumption 2.1.** Assume the following conditions hold.

(i) For each $i \in \mathbb{Z}_+$, there is a positive constant $L_i$ such that

$$|b(x, i) - b(y, i)| + |\sigma(x, i) - \sigma(y, i)| \leq L_i|x - y| \, \forall x, y \in \mathbb{R}^n.$$

(ii) $q_{ij}(\phi)$ is measurable in $\phi \in \mathcal{C}$ for all $i$ and $j \in \mathbb{Z}_+$. Moreover,

$$M := \sup_{\phi \in \mathcal{C}, i \in \mathbb{Z}_+} \{q_i(\phi)\} < \infty.$$

**Theorem 2.1.** *Under Assumption* 2.1*, for each initial data* $(\xi, i_0)$*, there exists a unique solution* $(X(t), \alpha(t))$ *to* (1.7)*.*

*Proof.* It is well-known that part (i) of Assumption 2.1 guarantees the existence and uniqueness of strong solutions to the following diffusion

$$dY(t) = b(Y(t), i)dt + \sigma(Y(t), i)dW(t) \quad \text{for each} \quad i \in \mathbb{Z}_+. \tag{2.1}$$

Then, given a stopping time $\tau$ and an $\mathcal{F}_\tau$-measurable $\mathbb{R}^n$-valued random variable $y = y(\tau)$ (depending on $\tau$), there exists a unique strong solution to (2.1) in $[\tau, \infty)$ satisfying $Y(\tau) = y(\tau)$ (see [31, Remark 3.10]). We can now construct the solution to (1.7) with initial data $(\xi, i_0)$ by the interlacing procedure similar to [3, Chapter 5]. Let $\tilde{Y}^{(0)}(t), t \geq 0$ be the solution with initial data $\xi(0)$ to

$$d\tilde{Y}^{(0)}(t) = b(\tilde{Y}^{(0)}(t), i_0)dt + \sigma(\tilde{Y}^{(0)}(t), i_0)dW(t).$$

We also set $\tilde{Y}^{(0)}(t) = \xi(t)$ for $t \in [-\tau, 0]$. Let

$$\begin{aligned} \tau_1 &= \inf\{t > 0 : \int_0^t \int_{\mathbb{R}} h(\tilde{Y}_s^{(0)}, i_0, z)\mathfrak{p}(ds, dz) \neq 0\} \quad \text{and} \\ i_1 &= i_0 + \int_0^{\tau_1} \int_{\mathbb{R}} h(\tilde{Y}_s^{(0)}, i_0, z)\mathfrak{p}(ds, dz), \end{aligned}$$

and $\tilde{Y}^{(1)}(t), t \geq \tau_1$ be the solution with $\tilde{Y}_{\tau_1}^{(1)} = \tilde{Y}_{\tau_1}^{(0)}$ to

$$d\tilde{Y}^{(1)}(t) = b(\tilde{Y}^{(1)}(t), i_1)dt + \sigma(\tilde{Y}^{(1)}(t), i_1)dW(t). \tag{2.2}$$

Define

$$\begin{aligned} \tau_2 &= \inf\{t > \tau_1 : \int_{\tau_1}^t \int_{\mathbb{R}} h(\tilde{Y}_s^{(1)}, i_1, z)\mathfrak{p}(ds, dz) \neq 0\} \quad \text{and} \\ i_2 &= i_1 + \int_{\tau_1}^{\tau_2} \int_{\mathbb{R}} h(\tilde{Y}_s^{(1)}, i_1, z)\mathfrak{p}(ds, dz). \end{aligned}$$

Note that, in the notation above, $\tilde{Y}_t^{(k)}$ is the function $s \in [-r, 0] \mapsto \tilde{Y}^{(k)}(t+s)$. Continuing this procedure, let $\tau_\infty = \lim_{k\to\infty} \tau_k$ and set

$$X(t) = \tilde{Y}^{(k)}(t), \ \ \alpha(t) = i_k \text{ if } \tau_k \le t < \tau_{k+1}. \tag{2.3}$$

Clearly, $X(t)$ satisfies that for every $t \ge 0$,

$$\begin{cases} X(t\wedge\tau_k) = X(0) + \displaystyle\int_0^{t\wedge\tau_k} \big[b(X(s),\alpha(s))ds + \sigma(X(s),\alpha(s))dW(t)\big] \\ \alpha(t\wedge\tau_k) = i_0 + \displaystyle\int_0^{t\wedge\tau_k}\int_{\mathbb{R}} h(X_s,\alpha(s-),z)\mathfrak{p}(ds,dz). \end{cases} \tag{2.4}$$

To show that $X(t)$ is a global solution, we need only prove that $\tau_\infty = \infty$ a.s. For any $T > 0$,

$$\begin{aligned} \mathbb{P}\{\tau_k \le T\} =& \mathbb{P}\Big\{\int_0^{T\wedge\tau_k}\int_{\mathbb{R}} \mathbf{1}_{\{z\in[0,q_{\alpha(s-)}(X_s))\}}\mathfrak{p}(ds,dz) = k\Big\} \\ \le& \mathbb{P}\Big\{\int_0^{T\wedge\tau_k}\int_{\mathbb{R}} \mathbf{1}_{\{z\in[0,M)\}}\mathfrak{p}(ds,dz) \ge k\Big\} \\ \le& \mathbb{P}\Big\{\int_0^{T}\int_{\mathbb{R}} \mathbf{1}_{\{z\in[0,M)\}}\mathfrak{p}(ds,dz) \ge k\Big\} \\ =& \sum_{l=k}^{\infty} e^{-MT}\frac{(MT)^l}{l!}. \end{aligned} \tag{2.5}$$

It follows that $\mathbb{P}\{\tau_k \le T\} \to 0$ as $k \to \infty$. As a result $\tau_\infty = \infty$ a.s. By this construction, it can be seen that $X(t)$ is continuous and $\alpha(t)$ is cadlag almost surely. The uniqueness of $(X(t),\alpha(t))$ follows from the uniqueness of $\tilde{Y}^{(k)}(t)$ on $[\tau_k, \tau_{k+1}]$ and the uniqueness of $i_k$ defined by

$$i_k = i_{k-1} + \int_{\tau_{k-1}}^{\tau_k}\int_{\mathbb{R}} h(\tilde{Y}_t^{(k-1)}, i_{k-1}, z)\mathfrak{p}(dt,dz).$$

This concludes the proof. □

**Assumption 2.2.** Assume the following conditions hold.

(i) There is a positive constant $L$ such that

$$|b(x,i)-b(y,i)|+|\sigma(x,i)-\sigma(y,i)|\leq L|x-y|,\ \forall x,y\in\mathbb{R}^n, i\in\mathbb{Z}_+.$$

(ii) $q_{ij}(\phi)$ is measurable in $\phi\in\mathcal{C}$ for each $(i,j)\in\mathbb{Z}_+^2$. Moreover, for any $H>0$,

$$M_H:=\sup_{\phi\in\mathcal{C},\|\phi\|\leq H,i\in\mathbb{Z}_+}\{q_i(\phi)\}<\infty.$$

**Remark 2.2.** We can use either Assumption 2.1 or Assumption 2.2 to obtain the existence and uniqueness of solutions to (1.7). Recall that now $\mathbb{Z}_+$ is a countable set, so care must be taken to distinct it with a finite state case. In Assumption 2.1, the Lipschitz constants of $b(\cdot,i),\sigma(\cdot,i)$ depend on $i$, and $q_i(\phi)$ is assumed to be bounded uniformly in $(\phi,i)\in\mathcal{C}\times\mathbb{Z}_+$. In contrast, the uniform boundedness of $q_i(\phi)$ is relaxed, but the Lipschitz constant of $b(\cdot,i),\sigma(\cdot,i)$ is assumed to be in $i\in\mathbb{Z}_+$.

**Theorem 2.3.** *Under Assumption* 2.2*, for each initial data* $(\xi,i_0)$*, there exists a unique solution* $(X(t),\alpha(t))$ *to* (1.7)*.*

*Proof.* Without loss of generality, we may assume that $(\xi,i_0)$ is bounded, since we can use the truncation method in [16, Theorem 3 in §6] to obtain the result for general $(\xi,i_0)$ once we have proved for the case $(\xi,i_0)$ being bounded. Construct the process $(X(t),\alpha(t))$ as in the proof of Theorem 2.1. We need to show that $\tau_\infty=\infty$ a.s. Following the proof of [32, Lemma 3.2, p. 51], there is a $K=K(T)$ such that

$$\mathbb{E}\Big(\sup_{0\leq t\leq T\wedge\tau_k}|X(t)|^2\Big)\leq K\,\forall\,k\in\mathbb{Z}_+.$$

As a result, for any $\varepsilon>0$, there is an $H_\varepsilon$ such that

$$\mathbb{P}\{\|X_t\|\leq H_\varepsilon\,\forall\,t\in[0,T\wedge\tau_k]\}>1-\frac{\varepsilon}{2}. \tag{2.6}$$

Let $\eta_{H_\varepsilon} = \inf\{t \geq 0 : \|X_t\| \geq H_\varepsilon\}$ and $M_{H_\varepsilon} = \sup_{\phi\in\mathcal{C},\|\phi\|\leq H_\varepsilon, i\in\mathbb{Z}_+}\{q_i(\phi)\} < \infty$. Then

$$\begin{aligned}\mathbb{P}\{\tau_k \leq T\wedge\eta_{H_\varepsilon}\} =&\mathbb{P}\Big\{\int_0^{T\wedge\tau_k\wedge\eta_{H_\varepsilon}}\int_{\mathbb{R}}\mathbf{1}_{\{z\in[0,q_{\alpha(s-)}(X_s))\}}\mathfrak{p}(ds,dz)=k\Big\}\\ \leq&\mathbb{P}\Big\{\int_0^{T\wedge\tau_k\wedge\eta_{H_\varepsilon}}\int_{\mathbb{R}}\mathbf{1}_{\{z\in[0,M_{H_\varepsilon})\}}\mathfrak{p}(ds,dz)\geq k\Big\}\\ \leq&\mathbb{P}\Big\{\int_0^{T}\int_{\mathbb{R}}\mathbf{1}_{\{z\in[0,M_{H_\varepsilon})\}}\mathfrak{p}(ds,dz)\geq k\Big\}\\ =&e^{-M_{H_\varepsilon}T}\sum_{l=k}^{\infty}\frac{(M_{H_\varepsilon}T)^l}{l!}.\end{aligned}\tag{2.7}$$

For sufficiently large $k$, we have

$$\mathbb{P}\{\tau_k \leq T\wedge\eta_{H_\varepsilon}\} \leq e^{-M_{H_\varepsilon}T}\sum_{l=k}^{\infty}\frac{(M_{H_\varepsilon}T)^l}{l!} \leq \frac{\varepsilon}{2}.\tag{2.8}$$

From (2.6) and (2.8), $\mathbb{P}\{\tau_k \geq T\} \geq \mathbb{P}(\{\tau_k\wedge T < \eta_{H_\varepsilon}\}\cap\{\tau_k > T\wedge\eta_{H_\varepsilon}\}) \geq 1-\varepsilon$ for sufficiently large $k$. Thus, we obtain that $\mathbb{P}\{\tau_\infty \geq T\} \geq 1-\varepsilon$. It holds for every $T > 0$ and $\varepsilon > 0$, so we obtain the desired result. □

**Remark 2.4.** To obtain the existence and uniqueness of solutions, Assumptions 2.1 and 2.2 can be relaxed by replacing the global Lipschitz conditions with local Lipschitz conditions together with Lyapunov-type functions. To be specific, let $V(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}$ be twice continuously differentiable in $x$. For each $i \in \mathbb{Z}_+$, let $\mathcal{L}_i V(x) = \nabla V(x) b(x,i) + \frac{1}{2}\operatorname{tr}\Big(\nabla^2 V(x) A(x,i)\Big)$. For instance (1) of Assumption 2.1 and (1) of Assumption 2.2 can be replaced by the following Assumptions 2.3 and 2.4, respectively.

**Assumption 2.3.** Assume the following conditions hold.

(i) For each $H > 0$, $i \in \mathbb{Z}_+$, there is a positive constant $L_{H,i}$ such that

$$|b(x,i)-b(y,i)| + |\sigma(x,i)-\sigma(y,i)| \leq L_{H,i}|x-y|,\ \forall |x|,|y| \leq H, i\in\mathbb{Z}_+.$$

(ii) For each $i \in \mathbb{Z}_+$, there exist a twice continuously differentiable function $V_i(x)$ and a constant $C_i > 0$ such that

$$\lim_{R\to\infty}\Big(\inf\{V_i(x) : |x| \geq R\}\Big) = \infty \quad \text{and} \quad \mathcal{L}_i V_i(x) \leq C_i(1 + V_i(x))\,\forall\, x \in \mathbb{R}^n.$$

**Assumption 2.4.** Assume the following conditions hold.

(i) For each $H > 0$, $i \in \mathbb{Z}_+$, there is a positive constant $L_{H,i}$ such that

$$|b(x,i) - b(y,i)| + |\sigma(x,i) - \sigma(y,i)| \leq L_{H,i}|x - y| \;\; \forall |x|, |y| \leq H, i \in \mathbb{Z}_+.$$

(ii) There exist a twice continuously differentiable function $V(x)$ and a constant $C > 0$ independent of $i \in \mathbb{Z}_+$ such that

$$\lim_{R\to\infty}\Big(\inf_{|x|\geq R}\{V(x)\}\Big) = \infty \text{ and } \mathcal{L}_i V(x) \leq C(1 + V(x)),\ x \in \mathbb{R}^n, i \in \mathbb{Z}_+.$$

**Theorem 2.5.** *For given initial data* $(\xi, i_0)$, *there exists a unique solution* $(X(t), \alpha(t))$ *to* (1.7) *if either of the following conditions is satisfied*

- *Assumption* 2.3 *and* (ii) *of Assumption* 2.1,
- *Assumption* 2.4 *and* (ii) *of Assumption* 2.2.

*Proof.* It is well known that Assumption 2.3 guarantees the existence and uniqueness of solutions to (2.1). Hence, if (ii) in Assumption 2.1 is satisfied, we can prove the desired result by using the proof of Theorem 2.1. Now, suppose Assumption 2.4 and (ii) of Assumption 2.2 hold. Similar to the proof of Theorem 2.3, we can assume that $(\xi, i_0)$ is bounded. Consider $X(t)$ and define $\tau_k$ as in the proof of Theorem 2.1. Then $X(t)$ is the solution with initial data $(\xi, i_0)$ to (1.7) on $[0, T \wedge \tau_k)$ for any $T > 0, k \in \mathbb{Z}_+$. We have from the generalized Itô formula that

$$\mathbb{E}V(X(T\wedge\tau_k\wedge\eta_H)) = \mathbb{E}V(\xi(0),i_0) + \mathbb{E}\int_0^{T\wedge\tau_k\wedge\eta_H} \mathcal{L}_i V(X(t),\alpha(t-))dt$$
$$\leq \mathbb{E}V(\xi(0),i_0) + C\mathbb{E}\int_0^{T\wedge\tau_k\wedge\eta_H}(1+V(X(t))dt,$$

where $\eta_H = \inf\{t\geq 0 : |X(t)| > H\}$. Using the estimate above and the argument in [31, Theorem 3.19], we can show that

$$\mathbb{E}V(T\wedge\tau_k\wedge\eta_H) \leq K = K(\xi,T)\,\forall H>0, k\in\mathbb{Z}_+.$$

In view of the property $\lim_{R\to\infty}\Big(\inf\{V(x) : |x|\geq R\}\Big) = \infty$, for any $\varepsilon > 0$, there is $H_\varepsilon > 0$ such that

$$\mathbb{P}\{\eta_{H_\varepsilon} > T\wedge\tau_k\} > 1-\frac{\varepsilon}{2}\ \forall\, k\in\mathbb{Z}_+.$$

Then, proceeding similarly as in the proof of Theorem 2.3 yields the existence and uniqueness of solutions with initial data $(\xi, i_0)$ to (1.7). □

**Example 2.6.** (cont. of Example 1.2) We come back to Example 1.2. We want to show that $X(t) > 0$ for all $t\geq 0$ under certain conditions. To proceed, we can set $Y(t) = \ln X(t)$ to obtain

$$dY(t) = [a(\alpha(t) - \frac{\sigma^2(\alpha(t))}{2} - b(\alpha(t))\exp(Y(t))]dt + \sigma(\alpha(t))dW(t). \tag{2.9}$$

To demonstrate (1.11) and (1.12) has a unique solution with $X(t) > 0$ for all $t\geq 0$, it is equivalent to show that (2.9) and (1.12) has a strong solution on $[0,\infty)$. Let $V(y) = e^y + e^{-y}$.

By direct calculation,

$$\begin{aligned}\mathcal{L}_i V(y) =& b(i) + (\sigma^2(i) - a(i))e^{-y} + a(i)e^{y} - b(i)e^{2y}\\ \leq& c(i) + (\sigma^2(i) - a(i))V(y),\end{aligned}$$

where $c(i) = \max_{y\in\mathbb{R}}\{b(i) + (2a(i) - \sigma^2(i))e^{y} - b(i)e^{2y}\}$. Applying Theorem 3.3, we can see that the equation has a unique solution if one of the following is satisfied

- $\beta_i(\phi) + \delta_i(\phi)$ is bounded uniformly in $\phi \in \mathcal{C}_+ := \{\psi \in \mathcal{C} : \psi(t) > 0\, \forall t \in [-r, 0]\}$ and $i \in \mathbb{Z}_+$.
- $c(i)$ and $\sigma^2(i) - a(i)$ are bounded above uniformly and for each $i \in \mathbb{Z}_+$, $\beta_i(\phi) + \delta_i(\phi)$ is bounded in each compact subset of $\phi \in \mathcal{C}_+$.

It can be shown by applying the result of the next section that the process $(Y_t, \alpha(t))$ has the Markov-Feller property if $\beta_i(\cdot)$ and $\delta_i(\cdot)$ are continuous in addition to one of the above conditions.

**Example 2.7.** (cont. of Example 1.3) To study the long-run average control problem in Example 1.3, it is important to make sure that the system under consideration processes ergodicity. Before the ergodicity can be verified, we need (1.14) has a unique solution for each initial condition. Denote the control set by $\widetilde{K}$ and assume it is a compact and convex set. Using a relaxed control representation $m_t(\cdot)$ (see [29]) to represent the consumption rate $c(\cdot)$, we can rewrite (1.14) as

$$dX(t) = \Big[\int_{\widetilde{K}} c(u)m_t(du) - \rho(\alpha(t))X(t)\Big]dt + \sigma(X(t), \alpha(t))dW(t). \tag{2.10}$$

Assume that for each $i \in \mathbb{Z}_+$, $\sigma(x, i)$ satisfies the conditions in Assumption 2.1 (i), and $Q(\phi)$ satisfies Assumption 2.1 (ii). Then the conditions of Theorem 2.1 are all verified. As a result, (2.10) has a unique solution for each initial condition.

## 2.2 Markov and Feller Properties

This section establishes the Markov and Feller properties of the process $(X_t, \alpha(t))$. While the Markov property can be derived by the well-known arguments, it requires much more efforts to obtain the Feller property. As already seen in the previous section, the past dependence and the use of $\mathbb{Z}_+$ make the analysis more complex than that of the switching diffusions with diffusion-dependent switching living in a finite set. To overcome the difficulties, in this section, we carry out the analysis by introducing some auxiliary or intermediate processes. First, it would be better if we could untangle the past dependence of the switching process and the infinity of the cardinality of its state space. For this purpose, we introduce a continuous-time Markov chain independent of the past and continuous state; we call this process $\gamma(t)$. Then naturally, associated with $\gamma(t)$, we examine a pair of process $(Z(t), \gamma(t))$. Even after this introduction, in the analysis, we still need to look into the details of the switching process $\alpha(t)$ such as when it jumps and the post jump location etc. To do so, we introduce another auxiliary process $Y(t)$, which is a "fixed"-$i$ process. We then have another pair of processes $(Y(t), \beta(t))$ to deal with. These auxiliary processes help us to establish the desired results. Their connections and interactions will be further specified in what follows.

First, note that the Brownian motion and the Poisson point process associated to $\mathfrak{p}(dt, dz)$ possess stationary strong Markov property, that is, for any finite stopping time $\eta$, $\{W^*(t)\}_{t\geq 0} = \{W(t+\eta) - W(\eta)\}_{t\geq 0}$ is an $\mathcal{F}_t^*$-Brownian motion and $\mathfrak{p}^*([t, t+s) \times U) = \mathfrak{p}([t+\eta, t+s+\eta) \times U)$ is a Poisson random measure with density $dt \times \mathfrak{m}(dz)$ (see [48, Theorem

101]). Hence, by standard arguments, we can obtain the following theorem whose proof is omitted. In fact, the theorem can be proved essentially by imitating the proof in [42, Chap. 5], [3, Chap. 6], or [40, Chap. 7].

**Theorem 2.8.** *Assume that the hypotheses of Theorem* 2.1*, or Theorem* 2.3*, or Theorem* 2.5 *are satisfied. Let* $(X(t), \alpha(t))$ *be a solution to* (1.7)*. Then* $(X_t, \alpha(t))$ *is a homogeneous strong Markov process taking value in* $\mathcal{C} \times \mathbb{Z}_+$ *with transition probabilities*

$$P(\phi, i, t, A \times \{j\}) = \mathbb{P}\{X_t^{\phi,i} \in A, \alpha(t) = j\},$$

*where* $X^{\phi,i}(t)$ *is the solution to* (1.7) *with initial data* $(\phi, i) \in \mathcal{C} \times \mathbb{Z}_+$*.*

We proceed with obtaining the Feller property of $(X_t, \alpha(t))$. Assuming that the hypotheses of Theorem 2.1, or Theorem 2.3, or Theorem 2.5 are satisfied leads to the existence and uniqueness of strong solutions. Next, we introduce an auxiliary hybrid diffusion with Markov switching. Let $\gamma^i(t)$ be a Markov chain starting at $i$ with generator $\tilde{Q} = (\rho_{ij})$ for $(i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, where $\rho_{ii} = -1$ and $\rho_{ij} = 2^{-j}$ if $j < i$ and $\rho_{ij} = 2^{-j+1}$ if $j > i$, that is,

$$\tilde{Q} = \begin{pmatrix} -1 & 1/2 & 1/4 & \cdots \\ 1/2 & -1 & 1/4 & \cdots \\ 1/2 & 1/4 & -1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We recursively define a sequence of stopping times $\{\theta_k^i\}$ with $\theta_k^i$ being the first jump time of $\gamma^i(t)$ after $\theta_{k-1}^i$ as follows

$$\theta_0^i = 0,\ \theta_k^i = \inf\{t > \theta_{k-1}^i : \gamma^i(t) \neq \gamma^i(\theta_{k-1}^i)\}, k \in \mathbb{Z}_+.$$

For $(\phi, i) \in \mathcal{C} \times \mathbb{Z}_+$, let $Z^{\phi,i}(t)$ be the solution to

$$dZ(t) = b(Z(t), \gamma(t))dt + \sigma(Z(t), \gamma(t))dW(t)\,, t \geq 0$$

satisfying $Z^{\phi,i}(t) = \phi(t)$ in $[-r, 0]$ and $\gamma(0) = i$.

Similar to Girsanov's theorem, which tells us how to convert an Itô process to a Brownian motion under a change of measure, we aim to establish a change of measure allowing us to "convert" a hybrid diffusion with past-dependent switching to a hybrid diffusion with Markov switching. To establish such a change of measure, we need to find the distribution of jump times of $\alpha(t)$. Because of the interactions between $\alpha(t)$ and $X(t)$, we need to introduce another auxiliary (or intermediate) process, which helps to examine the distribution of the jump times of $\alpha(t)$. Let $(Y^{\phi,i}(t), \beta^{\phi,i}(t))$ be the solution to

$$\begin{cases} dY(t) = b(Y(t), i)dt + \sigma(Y(t), i)dW(t), \;\; t \geq 0 \\ d\beta(t) = \displaystyle\int_{\mathbb{R}} h(Y_t, \beta(t-), z)\mathfrak{p}(dt, dz), t \geq 0 \end{cases} \tag{2.11}$$

satisfying $Y^{\phi,i}(t) = \phi(t)$ in $[-r, 0]$ and $\beta^{\phi,i}(0) = i$. By the definition, $\alpha^{\phi,i}(t) = \beta^{\phi,i}(t), X^{\phi,i}(t) = Y^{\phi,i}(t)$ up to the first jump time of the two process $\alpha(t)$ and $\beta(t)$. There is an advantage working with $(Y^{\phi,i}(t), \beta^{\phi,i}(t))$. Unlike the pair $(X(t), \alpha(t))$ in which $\alpha(t)$ depends on the continuous state, the process $Y^{\phi,i}(t)$ evolving for a fixed discrete state $i$ that does not depend on $\beta^{\phi,i}(t)$. Thus, it is easier to examine, for example, the first jump time of $\beta^{\phi,i}(t)$ (or $\alpha^{\phi,i}(t)$).

Next we recursively define sequences of stopping times associated with $\beta(t)$ and $\alpha(t)$ so that $\lambda_k^{\phi,i}$ and $\tau_k^{\phi,i}$ are the first jump times of the processes $\beta^{\phi,i}(t)$ and $\alpha^{\phi,i}(t)$ after $\lambda_{k-1}^{\phi,i}$ and

$\tau_{k-1}^{\phi,i}$, respectively. More specifically, for $k \in \mathbb{Z}_+$, let

$$\lambda_0^{\phi,i} = 0, \lambda_k^{\phi,i} = \inf\{t > \lambda_{k-1}^{\phi,i} : \beta^{\phi,i}(t) \neq \beta^{\phi,i}(\lambda_{k-1}^{\phi,i})\},\ i \in \mathbb{Z}_+.$$

and

$$\tau_0^{\phi,i} = 0,\ \tau_k^{\phi,i} = \inf\{t > \tau_{k-1}^{\phi,i} : \alpha^{\phi,i}(t) \neq \alpha^{\phi,i}(\tau_{k-1}^{\phi,i})\},\ i \in \mathbb{Z}_+.$$

To simplify the notation, we put

$$\alpha_k^{\phi,i} := \alpha^{\phi,i}(\tau_k^{\phi,i}),\ \beta_k^{\phi,i} := \beta^{\phi,i}(\lambda_k^{\phi,i}),\ \gamma_k^i := \gamma^i(\theta_k^i),$$

and

$$X_{(k)}^{\phi,i} := X_{\tau_k^{\phi,i}}^{\phi,i},\ Y_{(k)}^{\phi,i} := Y_{\lambda_k^{\phi,i}}^{\phi,i},\ Z_{(k)}^{\phi,i} := Z_{\theta_k^i}^{\phi,i},$$

where we use the subscript $k$ with parentheses to avoid confusion with the function-valued processes $X_t^{\phi,i}, Y_t^{\phi,i}, Z_t^{\phi,i}$ at $t = k$.

**Lemma 2.9.** *Let $g : \mathcal{C} \times \mathbb{R}_+ \times \mathbb{Z}_+ \mapsto \mathbb{R}$ be a bounded and measurable function, and $\mathcal{F}_T^W$ be the $\sigma$-algebra generated by $\{W(t), t \in [0,T]\}$. The following assertions hold:*

(i) $\mathbb{P}\big(\{\lambda_1^{\phi,i} > t\}\big|\mathcal{F}_T^W\big) = \mathbb{E}\Big[\mathbf{1}_{\{\lambda_1^{\phi,i}>t\}}\Big|\mathcal{F}_T^W\Big] = \exp\big(-\int_0^t q_i(Y_s^{\phi,i})ds\big)\ \ \forall\, t \in [0,T].$

(ii) $\mathbb{E}\Big[g(Y_{(1)}^{\phi,i}, \lambda_1^{\phi,i}, \beta_1^{\phi,i})\mathbf{1}_{\{\lambda_1^{\phi,i}\le T\}}\Big|\mathcal{F}_T^W\Big] = \sum_{j=1,j\neq i}^{\infty} \int_0^T g(Y_t, t, j) q_{ij}(Y_t)\exp(-\int_0^t q_i(Y_s)ds)dt.$

As indicated previously, it is difficult to estimate the difference of $X_t^{\phi_1,i}$ and $X_t^{\phi_2,i}$ because the states of $\alpha^{\phi_1,i}(t)$ and $\alpha^{\phi_2,i}(t)$ may differ significantly due to the continuous state dependence. In contrast, it is considerably easier to compare $Z_t^{\phi_1,i}$ and $Z_t^{\phi_2,i}$ because of the continuous-state-dependent switching is replaced by the continuous-state-independent

Markov chain. With help of the intermediate process $(Y(t), \beta(t))$ and Lemma 2.9, we obtain the following change of measure formula, which is a bridge to connect the continuous-state-dependent and continuous-state-independent processes.

**Proposition 2.10.** *For any $T > 0$, let $f(\cdot,\cdot) : \mathcal{C}\times\mathbb{Z}_+ \mapsto \mathbb{R}$ be a bounded continuous function. For any $l = 0, 1, \dots,$ any $i_k \in \mathbb{Z}_+$ with $i_k \neq i_{k+1}$ and $k = 1, \dots, l+1$, and any $(\phi, i) \in \mathcal{C}\times\mathbb{Z}_+$,*

$$\begin{aligned}
&\mathbb{E}\Big[f(X_T^{\phi,i}, \alpha^{\phi,i}(T))\mathbf{1}_{\{\tau_l^{\phi,i}\le T<\tau_{l+1}^{\phi,i}\}}\prod_{k=1}^{l}\mathbf{1}_{\{\alpha_k^{\phi,i}=i_k\}}\Big]\\
&=\exp(T)\mathbb{E}\Big[f(Z_T^{\phi,i}, i_l)\mathbf{1}_{\{\theta_l^i\le T<\theta_{l+1}^i\}}\prod_{k=1}^{l}\Big(\mathbf{1}_{\{\gamma_k^i=i_k\}}\frac{q_{i_k i_{k+1}}(Z_{(k)}^{\phi,i})}{\rho_{i_k i_{k+1}}}\Big)\times\\
&\qquad\times\exp\Big\{-\int_0^T q_{\gamma^i(s)}(Z_s^{\phi,i})ds\Big\}\Big].
\end{aligned}\tag{2.12}$$

**Remark 2.11.** The proofs of Lemma 2.9 and Proposition 2.10 will be given in the Appendix.

We are now in a position to prove the Feller property for the solution to (1.7). In addition to the sufficient conditions for the existence and uniqueness of solution, we prove the Feller property of the solution only with an additional condition that $q_{ij}(\phi)$ is continuous in $\phi$ for any $i, j \in \mathbb{Z}_+$. There are some difficulties because the process $\{X_t\}$ takes value in an infinite dimensional Banach space and the switching $\{\alpha(t)\}$ has an infinite state space. Moreover, although we suppose that $q_{ij}(\phi)$ is continuous, neither the uniform continuity in $\phi \in \mathcal{C}$ nor equi-continuity in $i, j \in \mathbb{Z}_+$ is assumed. Because of these difficulties, we divide the proof into several steps. First, we make the following assumptions, which will be relaxed later.

**Assumption 2.5.** Assume the following conditions hold.

(i) For each $i \in \mathbb{Z}_+$, $b(x, i)$ and $\sigma(x, i)$ are Lipschitz continuous functions that are vanishing outside $\{x : |x| \le R\}$ for some $R > 0$.

(ii) $M := \sup\{q_i(\phi) : i \in \mathbb{Z}_+, \phi \in \mathcal{C}\} < \infty$.

(iii) For each $i, j \in \mathbb{Z}_+, j \neq i$, $q_i(\cdot)$ and $q_{ij}(\cdot)$ are continuous on $\mathcal{C}$.

Before applying (2.12) to prove the continuous dependence of $u_f(\phi, i) = \mathbb{E}_{\phi,i} f(X_T, \alpha(T))$ on $(\phi, i)$, we first need the following lemma.

**Lemma 2.12.** *Assume that Assumption* 2.5 *is satisfied. Let* $(\phi_0, i_0) \in \mathcal{C} \times \mathbb{Z}_+$ *with* $\|\phi_0\| \le R$ *and* $T > 0$. *For each* $\Delta > 0$, *there exist* $m = m(\Delta) \in \mathbb{Z}_+$, $n_m = n_m(\Delta) \in \mathbb{Z}_+$, *and* $d_m = d_m(\Delta) > 0$ *such that*

$$\mathbb{P}\Big(\{\tau_{m+1}^{\phi,i_0} > T\} \cap \{\alpha^{\phi,i_0}(t) \in N_{n_m}, \forall t \in [0,T]\}\Big) \ge 1 - \Delta, \quad \forall \|\phi - \phi_0\| < d_m,$$

*where* $N_k = \{1, \dots, k\}$.

This lemma allows us to confine our attention to a finite subset of $\mathbb{Z}_+$ (the state space of $\alpha^{\phi,i_0}(\cdot)$) and a finite number of jumps when $\phi$ is close to $\phi_0$. It is a crucial step in providing some uniform estimates because we do not assume the equi-continuity of $q_{ij}(\cdot)$ in either $i$ or $j$. Since the switching intensity of $\alpha^{\phi,i_0}(t)$ depends on $X_t^{\phi,i_0}$, in order to obtain Lemma 2.12, we need to show that with an arbitrarily large probability, $X_t^{\phi,i_0}, t \in [0,T]$ belongs to a compact set in $\mathcal{C}$ for any $\phi$ sufficiently close to $\phi_0$. Note that under some suitable conditions, sample paths of a diffusion process in a finite interval $[0,T]$ are Hölder continuous. Thus, it is easy to find a compact set in which sample paths of a diffusion process lie with a large probability. Our arguments rely on this fact. However, the initial data $\phi$ of our process $X(t)$ does not always satisfy the Hölder continuity. Moreover, $X(t)$ depends on the state of $\alpha(t)$. We therefore need to introduce the following operator, which is motivated by merging trajectories of $X(t)$ at jump times. For $\mathcal{A}, \mathcal{B} \subset \mathcal{C}$, we define the set of continuous functions that are formed by merging functions in $\mathcal{A}$ and $\mathcal{B}$ as follows.

$$\mathcal{A} \uplus \mathcal{B} := \mathcal{A} \cup \mathcal{B} \cup \{\psi \in \mathcal{C} : \exists \psi_1 \in A, \psi_2 \in B, s \in [0, r] \text{ such that}$$
$$\psi(t) = \psi_1(s+t) \forall t \in [-r, -s]; \psi(t) = \psi_2(t+s-r) \forall t \in [-s, 0]\}.$$

By virtue of the Arzelá-Ascoli theorem, if $\mathcal{A}$ and $\mathcal{B}$ are compact, so is $\mathcal{A} \uplus \mathcal{B}$. Using this fact and the Hölder continuity of sample paths of a diffusion process, we can find a suitable compact set to which $X_t^{\phi,i_0}, t \in [0, T]$, belongs with a large probability for any $\phi$ which is sufficiently close to $\phi_0$. Then, Lemma 2.12 can be proved. The details of the proof are postponed to the Appendix. Now, we point out some nice properties of the diffusion process with Markov switching $(Z(t), \gamma(t))$, which are useful to compare the sample paths of $Z(t)$ with different initial values.

**Lemma 2.13.** *Fix $i_0 \in \mathbb{Z}_+$. For each $k \in \mathbb{Z}_+$ and $\varepsilon > 0$, there is an $\hbar_k = \hbar_k(\varepsilon) > 0$ such that*

$$\mathbb{P}\Big\{ \sup_{t,s \in [0, T\wedge \iota_k], 0 < t-s < \hbar_k} \frac{|Z^{\phi,i_0}(t) - Z^{\phi,i_0}(s)|}{(s-t)^{0.25}} \le 4 \Big\} > 1 - \varepsilon \,\forall\, |\phi(0)| \le R,$$

*and*

$$\mathbb{E}\Big[ \sup_{t \in [0, T\wedge\iota_k]} |Z^{\phi,i_0} - Z^{\psi,i_0}|^2 \Big] \le \overline{C} |\phi - \psi|^2,$$

*where $\iota_k = \inf\{t > 0 : \gamma^{i_0}(t) > k\}$ and $\overline{C}$ is some positive constant.*

*Proof.* Since $b(x, i)$ and $\sigma(x, i)$ are Lipschitzian in $x$ uniformly in $N_k$, by standard arguments (e.g., [31, Theorem 3.23]), we can show that

$$\mathbb{E}|x(t \wedge \iota_k) - x(s \wedge \iota_k)|^6 < \tilde{C}_k (t-s)^3, \forall 0 \le s \le t \le T.$$

Using the Kolmogorov-Centsov theorem, we obtain the first inequality. The details are similar to the proof of Lemma 2.12 in the Appendix. The second claim is proved in the same manner as that of [60, Lemma 2.14]. □

Having Lemmas 2.12 and 2.13, we are ready to use the change of measure (2.12) to prove the Feller property of $(X_t, \alpha(t))$ under Assumption 2.5.

**Proposition 2.14.** *Suppose that Assumption 2.5 is satisfied. Let $f(\cdot,\cdot) : \mathcal{C} \times \mathbb{Z}_+ \mapsto \mathbb{R}$ be continuous and bounded. Then for any $T > 0$, $u_f(\phi, i) = \mathbb{E}f(X_T^{\phi,i}, \alpha^{\phi,i}(T))$ is a continuous function in $\phi \in \mathcal{C}$.*

*Proof.* We suppose without loss of generality that $|f(\phi, i)| \le 1 \,\forall\, (\phi, i) \in \mathcal{C}\times\mathbb{Z}_+$. Fix $(\phi_0, i_0) \in \mathcal{C} \times \mathbb{Z}_+$. We show that for any $\Delta > 0$, there exists $d^* = d^*(\Delta, \phi_0, i_0) > 0$ such that

$$\left|\mathbb{E}f(X_T^{\phi,i_0}, \alpha^{\phi,i_0}(T)) - \mathbb{E}f(X_T^{\phi_0,i_0}, \alpha^{\phi_0,i_0}(T))\right| \le 3\Delta \,\forall\, \|\phi - \phi_0\| < d^*. \tag{2.13}$$

In view of Lemma 2.12, there are $m$, $n_m \in \mathbb{Z}_+$, and $d_m > 0$ such that

$$\mathbb{P}\Big(\{\tau_{m+1}^{\phi,i_0} > T\} \cap \{\alpha^{\phi,i_0}(t) \in N_{n_m}, \forall t \in [0, T]\}\Big) \ge 1 - \Delta \,\forall \|\phi - \phi_0\| < \frac{d_m}{2}. \tag{2.14}$$

Let $\varepsilon = \varepsilon(\Delta) > 0$ (to be specified later). Let $\hbar_k$ be as in Lemma 2.13. Denote

$$\tilde{\mathcal{H}} = \left\{\psi(\cdot) \in \mathcal{C} : \|\psi\| \le R + 1 \text{ and } \sup_{t,s\in[-r,0],0<t-s<\hbar_{n_m}} \frac{|\psi(s) - \psi(t)|}{(s - t)^{0.25}} \le 4\right\}$$

and $\tilde{\mathcal{K}} = \{\phi_0\} \uplus \tilde{\mathcal{H}}$. By the compactness of $\tilde{K}$, there is a $\tilde{d}_m > 0$ such that

$$\|q_{ij}(\psi) - q_{ij}(\phi)\| < \varepsilon, \; |f(\psi, i) - f(\phi, i)| < \varepsilon \tag{2.15}$$

if $\phi \in \tilde{\mathcal{K}}, i, j \in N_{n_m}$ and $\|\psi - \phi\| < \tilde{d}_m$. In view of Lemma 2.13, we can choose $\widehat{d}_m > 0$ such that

$$\mathbb{P}\Big\{\sup_{t\in[0,T\wedge\iota_k]}\|Z_t^{\phi,i_0}-Z_t^{\phi_0,i_0}\|\le\tilde{d}_m\Big\}<\varepsilon \text{ if } \|\phi-\phi_0\|\le\widehat{d}_m. \tag{2.16}$$

Let $A^\phi$ be the event $\{\tau_m^{\phi,i_0}\le T<\tau_{m+1}^{\phi,i_0},\iota_{n_m}>T\}$ and $l(T)$ be the number of jumps up to time $T$ of $\gamma^{i_0}(t)$. It follows from Proposition 2.10 that

$$\begin{aligned}&\mathbb{E}\big[f(X_T^{\phi,i_0},\alpha^{\phi,i_0}(T))\mathbf{1}_{A^\phi}\big]-\mathbb{E}\big[f(X_T^{\phi_0,i_0},\alpha^{\phi_0,i_0}(T))\mathbf{1}_{A^{\phi_0}}\big]\\&=\exp(T)\mathbb{E}\Big[\mathbf{1}_{\{l(T)<m+1,\iota_{n_m}>T\}}\Big[g(Z^{\phi,i_0}(\cdot),\gamma^{i_0}(\cdot))-g(Z^{\phi_0,i_0}(\cdot),\gamma^{i_0}(\cdot))\Big]\Big],\end{aligned} \tag{2.17}$$

where

$$g(Z^{\phi,i_0}(\cdot),\gamma^{i_0}(\cdot))=f(Z_T^{\phi,i_0},\gamma^{i_0}(T))\prod_{k=1}^{l(T)}\frac{q_{\gamma_k^{i_0}\gamma_{k+1}^{i_0}}(Z_{(k+1)}^{\phi,i_0})}{\rho_{\gamma_k^{i_0}\gamma_{k+1}^{i_0}}}e^{-\int_0^T q_{\gamma^{i_0}(s)}(Z_s^{\phi,i_0})ds}.$$

Let $D_m^\phi$ be the event

$$\begin{aligned}D_m^\phi:=&\Big\{\sup_{t\in[0,T\wedge\iota_k]}\|Z_t^{\phi,i_0}-Z_t^{\phi_0,i_0}\|\le\tilde{d}_m\Big\}\\&\cap\Big\{\sup_{t,s\in[0,T\wedge\iota_k],0<t-s<\hbar_{n_m}}\frac{|Z^{\phi_0,i_0}(s)-Z^{\phi_0,i_0}(t)|}{(s-t)^{0.25}}\le 4\Big\}\end{aligned}$$

Using (2.17) and the estimates in [60, Lemma 2.17], we obtain for $l \geq 1$,

$$\begin{aligned}
&\Big|\mathbb{E}\big[f(X_T^{\phi,i_0},\alpha^{\phi,i_0}(T))\mathbf{1}_{A_l^\phi}\big]-\mathbb{E}\big[f(X_T^{\phi_0,i_0},\alpha^{\phi_0,i_0}(T))\mathbf{1}_{A_l^{\phi_0}}\big]\Big|\\
\leq& K\mathbb{E}\Big[\mathbf{1}_{\{\theta_l^{i_0}\leq T<\theta_{l+1}^{i_0},\iota_{n_m}>T\}}\times\sup_{i\in N_{n_m}}|f(Z_T^{\phi,i_0},i)-f(Z_T^{\phi_0,i_0},i)|\Big]\\
&+K\mathbb{E}\Big[\mathbf{1}_{\{\theta_l^{i_0}\leq T<\theta_{l+1}^{i_0},\iota_{n_m}>T\}}\times\sup_{t\in[0,T],i,j\in N_{n_m}}|q_{ij}(Z_t^{\phi,i_0})-q_{ij}(Z_t^{\phi_0,i_0})|\Big]\\
\leq& K\mathbb{E}\Big[\mathbf{1}_{\{\theta_l^{i_0}\leq T<\theta_{l+1}^{i_0},\iota_{n_m}>T\}}\mathbf{1}_{D_m^\phi}\times\sup_{i\in N_{n_m}}|f(Z_T^{\phi,i_0},i)-f(Z_T^{\phi_0,i_0},i)|\Big]\\
&+K\mathbb{E}\Big[\mathbf{1}_{\{\theta_l^{i_0}\leq T<\theta_{l+1}^{i_0},\iota_{n_m}>T\}}\mathbf{1}_{D_m^\phi}\times\sup_{t\in[0,T],i,j\in N_{n_m}}|q_{ij}(Z_t^{\phi,i_0})-q_{ij}(Z_t^{\phi_0,i_0})|\Big]\\
&+2K(M+1)\mathbb{P}(\Omega\setminus D_m^\phi),
\end{aligned}$$

where $K$ is a constant depending only on $T, m, n_m$.

Note that if $\omega \in \{\theta_l^{i_0} \leq T < \theta_{l+1}^{i_0}, \iota_{n_m} > T\} \cap D_m^\phi$, then $Z_t^{\phi_0,i_0} \in \tilde{\mathcal{K}}$ and $\|Z_t^{\phi,i_0} - Z_t^{\phi_0,i_0}\| \leq \tilde{d}_m \, \forall t \in [0,T]$ which implies in view of (2.15) that

$$\sup_{i\in N_{n_m}}|f(Z_T^{\phi,i_0},i)-f(Z_T^{\phi_0,i_0},i)|+\sup_{t\in[0,T],i,j\in N_{n_m}}|q_{ij}(Z_t^{\phi,i_0})-q_{ij}(Z_t^{\phi_0,i_0})|<2\varepsilon.$$

On the other hand, Lemma 2.13 and (2.16) imply that

$$\mathbb{P}(\Omega\setminus D_m^\phi)\leq 3\varepsilon \ \text{ if } \ \|\phi-\phi_0\|\leq\widehat{d}_m.$$

Hence for $\|\phi - \phi_0\| \leq \widehat{d}_m$, we have that

$$\Big|\mathbb{E}\big[f(X_T^{\phi,i_0},\alpha^{\phi,i_0}(T))\mathbf{1}_{A^\phi}\big]-\mathbb{E}\big[f(X_T^{\phi_0,i_0},\alpha^{\phi_0,i_0}(T))\mathbf{1}_{A^{\phi_0}}\big]\Big|\leq 2K(4+3M)\varepsilon, \tag{2.18}$$

Note that

$$\mathbb{P}\big(\Omega\setminus A^\phi\big)=\mathbb{P}\big(\{\tau_{m+1}^{\phi,i_0}<T\}\cup\{\alpha^{\phi,i_0}(t)\notin N_{n_m}\text{ for some }t\in[0,T]\}\big)<\Delta,$$

which implies

$$\left|\mathbb{E}\big[f(X_T^{\phi,i_0},\alpha^{\phi,i_0}(T))\mathbf{1}_{\Omega\setminus A^\phi}\big]-\mathbb{E}\big[f(X_T^{\phi_0,i_0},\alpha^{\phi_0,i_0}(T))\mathbf{1}_{\Omega\setminus A^\phi}\big]\right|\le 2\Delta. \tag{2.19}$$

Choosing $\varepsilon=\dfrac{\Delta}{2K(4+3M)}$, we have from (2.18) and (2.19) that

$$\left|\mathbb{E}\big[f(X_T^{\phi,i_0},\alpha^{\phi,i_0}(T))\big]-\mathbb{E}\big[f(X_T^{\phi_0,i_0},\alpha^{\phi_0,i_0}(T))\big]\right|\le 3\Delta$$

if $\|\phi-\phi_0\|<d^*:=\dfrac{d_m}{2}\wedge\widehat{d}_m.$ □

With the above technical preparations, we are now in a position to prove the main theorem of this section. By using truncation arguments, we can obtain the Feller property of $(X_t,\alpha(t))$ even if $b(\cdot,i),\sigma(\cdot,i)$ do not vanish outside a bounded region and $q_i(\phi)$ is not bounded. The precise condition is given below.

**Theorem 2.15.** *Assume that the hypotheses of Theorem* 2.1*, or Theorem* 2.3*, or Theorem* 2.5 *hold. Assume further that $q_{ij}(\cdot)$ is a continuous function for any $i,j\in\mathbb{Z}_+$. Then the solution to* (1.7) *has the Feller property.*

*Proof.* Let $f(\cdot,\cdot):\mathcal{C}\times\mathbb{Z}_+\mapsto\mathbb{R}$ be a continuous function with $|f(\phi,i)|\le 1\,\forall\,(\phi,i)\in\mathcal{C}\times\mathbb{Z}_+$. Fix $R>0,T>0$. Suppose that $\|\phi_0\|<R$. Under the hypotheses of Theorem 2.1, or Theorem 2.3, or Theorem 2.5, it is shown in the proofs of those theorems that for any $\varepsilon>0$, there exists an $\tilde{R}>0$ such that

$$\mathbb{P}\{\|X_t^{\phi,i_0}\|\le\tilde{R}\}>1-\frac{\varepsilon}{8}\,\forall\|\phi\|\le R+1. \tag{2.20}$$

Let $\Phi(x):\mathbb{R}^n\mapsto\mathbb{R}$ be a twice continuously differentiable satisfying $\Phi(x)=1$ if $|x|\le\tilde{R}$ and

$\Phi(x) = 0$ if $|x| \geq \tilde{R} + 1$. Let $(\tilde{X}_t^{\phi,i_0}, \tilde{\alpha}^{\phi,i_0}(t))$ be the solution with initial data $(\phi, i_0)$ to

$$\begin{cases} d\tilde{X}(t) = \Phi(\tilde{X}(t))b(\tilde{X}(t), \tilde{\alpha}(t))dt + \Phi(\tilde{X}(t))\sigma(\tilde{X}(t), \tilde{\alpha}(t))dW(t) \\ d\tilde{\alpha}(t) = \displaystyle\int_{\mathbb{R}} h(\tilde{X}_t, \tilde{\alpha}(t-), z)\mathfrak{p}(dt, dz). \end{cases} \tag{2.21}$$

Then $(\tilde{X}^{\phi,i_0}(t), \tilde{\alpha}^{\phi,i_0}(t)) = (X^{\phi,i_0}(t), \alpha^{\phi,i_0}(t))$ up to the time that $\|X_t^{\phi,i_0}\| > \tilde{R}$, which combined with (2.20) implies

$$\mathbb{P}\{\tilde{\Omega}_{\phi,i_0}\} > 1 - \frac{\varepsilon}{8}, \ \forall \|\phi\| < R$$

where $\tilde{\Omega}_{\phi,i_0} := \{\tilde{X}_T^{\phi,i_0} = X_T^{\phi,i_0}, \tilde{\alpha}^{\phi,i_0}(T) = \alpha^{\phi,i_0}(T)\}$. As a result, if $\|\phi\| < R$, we have

$$\begin{aligned} &\left|\mathbb{E}f(X_T^{\phi,i_0}, \alpha^{\phi,i_0}(T)) - \mathbb{E}f(\tilde{X}_T^{\phi,i_0}, \tilde{\alpha}^{\phi,i_0}(T))\right| \\ &\quad\leq \mathbb{E}\left[\mathbf{1}_{\tilde{\Omega}^c_{\phi,i_0}}\left|f(X_T^{\phi,i_0}, \alpha^{\phi,i_0}(T)) - f(\tilde{X}_T^{\phi,i_0}, \tilde{\alpha}^{\phi,i_0}(T))\right|\right] \ (\text{with } \tilde{\Omega}^c_{\phi,i_0} = \Omega \setminus \tilde{\Omega}_{\phi,i_0}) \\ &\quad\leq 2\mathbb{P}\left(\mathbf{1}_{\tilde{\Omega}^c_{\phi,i_0}}\right) \leq 2\frac{\varepsilon}{8} = \frac{\varepsilon}{4}. \end{aligned} \tag{2.22}$$

It follows from Proposition 2.14 that there exists a $\delta \in (0,1)$ such that if $\|\phi - \phi_0\| < \delta$, we have

$$\left|\mathbb{E}f(\tilde{X}_T^{\phi,i_0}, \tilde{\alpha}^{\phi,i_0}(T)) - \mathbb{E}f(\tilde{X}_T^{\phi_0,i_0}, \tilde{\alpha}^{\phi_0,i_0}(T))\right| < \frac{\varepsilon}{2}. \tag{2.23}$$

Since $\|f\| \leq 1$, we can easily obtain from (2.22) and (2.23) that

$$\begin{aligned} &\left|\mathbb{E}f(X_T^{\phi,i_0}, \alpha^{\phi,i_0}(T)) - \mathbb{E}f(X_T^{\phi_0,i_0}, \alpha^{\phi_0,i_0}(T))\right| \\ &\leq \left|\mathbb{E}f(\tilde{X}_T^{\phi,i_0}, \tilde{\alpha}^{\phi,i_0}(T)) - \mathbb{E}f(\tilde{X}_T^{\phi_0,i_0}, \tilde{\alpha}^{\phi_0,i_0}(T))\right| \\ &\quad + \left|\mathbb{E}f(X_T^{\phi,i_0}, \alpha^{\phi,i_0}(T)) - \mathbb{E}f(\tilde{X}_T^{\phi,i_0}, \tilde{\alpha}^{\phi,i_0}(T))\right| \\ &\quad + \left|\mathbb{E}f(X_T^{\phi_0,i_0}, \alpha^{\phi_0,i_0}(T)) - \mathbb{E}f(\tilde{X}_T^{\phi_0,i_0}, \tilde{\alpha}^{\phi_0,i_0}(T))\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon, \ \text{ if } \ \|\phi - \phi_0\| < \delta. \end{aligned}$$

The proof of the theorem is complete. □

## 2.3 Feller Property of Hybrid Diffusion without Past Dependence

Now, we suppose that the $q_{ij}, i, j \in \mathbb{Z}_+$ associated with $\alpha(t)$ depend only the current state of $X(t)$. To be more precise $q_{ij}(\cdot)$ is a function from $\mathbb{R}^n$ to $\mathbb{R}$ for each $(i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+$. As a special case of the hybrid diffusion with past-dependent switching, we obtain the following theorem.

**Theorem 2.16.** *Assume that $q_{ij}(\cdot)$ is a continuous function for any $i, j \in \mathbb{Z}_+$. Assume further that one of the following conditions is satisfied:*

*(a) Assumption 2.3 and $q_i(y) = \sum_{j \neq i} q_{ij}(y)$ is bounded uniformly in $(i, y) \in \mathbb{Z}_+ \times \mathbb{R}^n$.*

*(b) Assumption 2.4 and $q_i(y)$ is bounded uniformly in $(i, y) \in \mathbb{Z}_+ \times K$ for each compact subset $K$ of $\mathbb{R}^n$.*

*Then the unique solution to (1.7) is a Markov process having the Feller property.*

**Remark 2.17.** If for each discrete state $i \in \mathbb{Z}_+$, the diffusion $Y^{(i)}(t)$, which is the solution process to

$$dY^{(i)}(t) = b(Y^{(i)}(t), i)dt + \sigma(Y^{(i)}(t), i)dW(t) \tag{2.24}$$

has the strong Feller property, we do not need the continuity of $q_{ij}(\cdot)$ to get the Feller property of $(X(t), \alpha(t))$. In fact, we will obtain a stronger result, namely, the strong Feller property. The condition for the strong Feller property of $Y^{(i)}(t)$ is essentially the ellipticity of $A(x, i)$ or the Hörmander condition for hyperellipticity (see e.g., [38, 50]).

**Theorem 2.18.** *Assume that $q_{ij}(\cdot)$ is measurable for any $i, j \in \mathbb{Z}_+$ and either* (A) *or* (B) *in Theorem* 2.16 *holds. If for each $i \in \mathbb{Z}_+$, the solution process $Y^{(i)}(t)$ to* (2.24) *has the* *strong Feller property, then the unique solution to* (1.7) *has the strong Feller property, that is, for each bounded measurable function* $g(y,i) : \mathbb{R}^n \times \mathbb{Z}_+ \to \mathbb{R}$, *the function* $(x,i) \to \mathbb{E}g(X^{x,i}(T), \alpha^{x,i}(T))$ *is continuous for each* $T > 0$.

*Proof.* We assume without loss of generality that $|g(z,i)| \le 1 \, \forall z \in \mathbb{R}^n, i \in \mathbb{Z}_+$. Let $Y^{y,i}(t)$ be the solution with initial data $y$ to (2.24). Fix $(x,i) \in \mathbb{R}^n \times \mathbb{Z}_+$ and $\varepsilon > 0$. Under the assumption $(A)$ or $(B)$, we can show that for each $x \in \mathbb{R}^n$, there is a $K > 0$ satisfying

$$\mathbb{P}((\Omega_\varepsilon^{y,i})^c) < \frac{\varepsilon}{8} \, \forall y \in B(x,1) := \{z : |x - z| < 1\}, \tag{2.25}$$

where $\Omega_\varepsilon^{y,i} = \{|Y^{y,i}(t)| \le K, t \in [0,1]\}$, $(\Omega_\varepsilon^{y,i})^c = \Omega \setminus \Omega_\varepsilon^{y,i}$.

Let $M = \sup_{i \in \mathbb{Z}_+, |z| \le K} q_i(z)$ and $t_0 \in (0,1)$ satisfying $1 - \exp\{-Mt_0\} < \dfrac{\varepsilon}{16}$. It follows from (2.25) and (i) of Lemma 2.9 that

$$\mathbb{P}\{\tau^{y,i} > t_0\} > 1 - \frac{3\varepsilon}{16} \, \forall y \in B(x,1), \tag{2.26}$$

where

$$\tau^{y,i} := \inf\{t > 0 : \alpha^{y,i}(t) \ne i\} = \inf\Big\{t > 0 : \int_0^t \int_{\mathbb{R}} h(Y^{y,i}(s), i, u)\mathfrak{p}(ds, du) \ne 0\Big\}.$$

Denote $\Phi(y,i) := \mathbb{E}g\big(X^{y,i}(T - t_0), \alpha^{y,i}(T - t_0)\big)$. The condition $|g(y,i)| \le 1$ implies $|\Phi(y,i)| \le 1$ for all $y \in \mathbb{R}^n, i \in \mathbb{Z}_+$. By the strong Feller property of $Y^{(i)}(t)$, there is a $\delta > 0$ such that

$$|\mathbb{E}\Phi(Y^{y,i}(t_0), i) - \mathbb{E}\Phi(Y^{x,i}(t_0), i)| \le \frac{\varepsilon}{4} \, \forall y \in B(x, \delta). \tag{2.27}$$

By the strong Markov property of $X(t)$, we have

$$\begin{aligned} \mathbb{E}g\big(X^{y,i}(T), \alpha^{y,i}(T)\big) &= \mathbb{E}\Phi(X^{y,i}(t_0), \alpha^{y,i}(t_0)) \\ &= \mathbb{E}\big[\mathbf{1}_{\{\tau^{y,i} > t_0\}}\Phi(X^{y,i}(t_0), \alpha^{y,i}(t_0))\big] + \mathbb{E}\big[\mathbf{1}_{\{\tau^{y,i} \le t_0\}}\Phi(X^{y,i}(t_0), \alpha^{y,i}(t_0))\big]. \end{aligned} \tag{2.28}$$

Applying (i) of Lemma 2.9, we obtain

$$\begin{aligned}
&\mathbb{E}\big[\mathbf{1}_{\{\tau^{y,i}>t_0\}}\Phi(X^{y,i}(t_0),\alpha^{y,i}(t_0))\big]\\
&=\mathbb{E}\big[\Phi(Y^{y,i}(t_0),\alpha^{y,i}(t_0))e^{-\int_0^{t_0}q_i(Y^{y,i}(s))ds}\big]\\
&=\mathbb{E}\big[\Phi(Y^{y,i}(t_0),\alpha^{y,i}(t_0))\big]+\mathbb{E}\Big[\mathbf{1}_{(\Omega_\varepsilon^{y,i})^c}\Phi(Y^{y,i}(t_0),\alpha^{y,i}(t_0))\Big(e^{-\int_0^{t_0}q_i(Y^{y,i}(s))ds}-1\Big)\Big]\\
&\quad+\mathbb{E}\Big[\mathbf{1}_{\Omega_\varepsilon^{y,i}}\Phi(Y^{y,i}(t_0),\alpha^{y,i}(t_0))\Big(e^{-\int_0^{t_0}q_i(Y^{y,i}(s))ds}-1\Big)\Big].
\end{aligned}\tag{2.29}$$

Note that if $|g(z,i)|\le 1\,\forall z\in\mathbb{R}^n, i\in\mathbb{Z}_+$ then $|\Phi(z,i)|\le 1\,\forall z\in\mathbb{R}^n, i\in\mathbb{Z}_+$. We have the following estimates for $y\in B(x,\delta)$ using (2.25), (2.26), (2.27), and the fact that $|g(z,i)|,|\Phi(z,i)|\le 1\,\forall z\in\mathbb{R}^n, i\in\mathbb{Z}_+$.

$$\Big|\mathbb{E}\big[\mathbf{1}_{(\Omega_\varepsilon^{y,i})^c}\Phi(Y^{y,i}(t_0),\alpha^{y,i}(t_0))\big(e^{-\int_0^{t_0}q_i(Y^{y,i}(s))ds}-1\big)\big]\Big|\le\mathbb{P}((\Omega_\varepsilon^{y,i})^c)\le\frac{\varepsilon}{8},\tag{2.30}$$

$$\Big|\mathbb{E}\Big[\mathbf{1}_{\Omega_\varepsilon^{y,i}}\Phi(Y^{y,i}(t_0),\alpha^{y,i}(t_0))\Big(e^{-\int_0^{t_0}q_i(Y^{y,i}(s))ds}-1\Big)\Big]\Big|\le 1-\exp(-Mt_0)\le\frac{\varepsilon}{16},\tag{2.31}$$

$$\mathbb{E}\big[\mathbf{1}_{\{\tau^{y,i}\le t_0\}}\Phi(X^{y,i}(t_0),\alpha^{y,i}(t_0))\big]\Big|\le\mathbb{P}\{\tau^{y,i}\le t_0\}\le\frac{3\varepsilon}{16}.\tag{2.32}$$

Applying estimates (2.27), (2.30), (2.31), and (2.32) to (2.28) and (2.29), we have

$$\Big|\mathbb{E}\Phi(X^{y,i}(T),\alpha^{y,i}(T))-\mathbb{E}\Phi(X^{x,i}(T),\alpha^{x,i}(T))\Big|\le\varepsilon,\ \forall y\in B(x,\delta).$$

The proof is complete. □

**Remark 2.19.** Sufficient conditions for the strong Feller property of $(X(t),\alpha(t))$, in which the rates of switching $q_{ij}$ for $i,j\in\mathbb{Z}_+$ depend only on the current continuous state $X(t)$, was obtained in Shao [44]. However, the conditions there are restrictive. To obtain the strong Feller property, it is assumed in [44] that $q_{ij}(x)$, $b(x,i)$ and $\sigma(x,i)$ are Lipschitz in $x$ uniformly in $i\in\mathbb{Z}_+$. The ellipticity of $A(x,i)$ is also assumed to be uniform in $(x,i)\in\mathbb{R}^n\times\mathbb{Z}_+$.

Moreover, it is assumed that $q_{ij}(x) = 0$ if $|i - j| < k$ for some constant $k$. It can be seen that our conditions in this paper are much more relaxed compared with the aforementioned reference.

# CHAPTER 3 RECURRENCE AND ERGODICITY

Throughout this chapter, we suppose that one of the assumptions for existence and uniqueness of solutions in Chapter 2 is satistifed, that is, any of the following holds

- Assumption 2.1,
- Assumption 2.2,
- Assumption 2.3 and (ii) of Assumption 2.1,
- Assumption 2.4 and (ii) of Assumption 2.2.

We also assume that $q_{i,j}(\phi)$ is continuous in $\mathcal{C}$ for each $(i,j) \in \mathbb{Z}^2_+$. To proceed, we need the functional Itô formula.

## 3.1 The Functional Itô Formula

Now we state the functional Itô formula for our process (see [10] for more details). Let $\mathbb{D}$ be the space of cadlag functions $f : [-r, 0] \mapsto \mathbb{R}^n$. For $\phi \in \mathbb{D}$, we define horizontal and vertical perturbations for $h \geq 0$ and $y \in \mathbb{R}^n$ as

$$\phi_h(s) = \begin{cases} \phi(s+h) \ \text{ if } \ s \in [-r, -h], \\ \phi(0) \ \text{ if } \ s \in [-h, -0], \end{cases}$$

and

$$\phi^y(s) = \begin{cases} \phi(s) \ \text{ if } \ s \in [-r, 0), \\ \phi(0) + y, \end{cases}$$

respectively. Let $V : \mathbb{D} \times \mathbb{Z}_+ \mapsto \mathbb{R}$. The horizontal derivative at $(\phi, i)$ and vertical partial derivative of $V$ are defined as

$$V_t(\phi, i) = \lim_{h \to 0} \frac{V(\phi_h, i) - V(\phi)}{h} \tag{3.1}$$

and

$$\partial_i V(\phi, i) = \lim_{h \to 0} \frac{V(\phi^{he_i}, i) - V(\phi)}{h} \tag{3.2}$$

if these limits exist. In (3.2), $e_i$ is the standard unit vector in $\mathbb{R}^n$ whose $i$-th component is 1 and other components are 0. Let $\mathbb{F}$ be the family of function $V(\cdot, \cdot) : \mathbb{D} \times \mathbb{Z}_+ \mapsto \mathbb{R}$ satisfying that

- $V$ is continuous, that is, for any $\varepsilon > 0$, $(\phi, i) \in \mathbb{D} \times \mathbb{Z}_+$, there is a $\delta > 0$ such that $|V(\phi, i) - V(\phi', i)| < \varepsilon$ as long as $\|\phi - \phi'\| < \delta$.
- The derivatives $V_t$, $V_x = (\partial_k V)$, and $V_{xx} = (\partial_{kl} V)$ exist and are continuous.
- $V$, $V_t$, $V_x = (\partial_k V)$ and $V_{xx} = (\partial_{kl} V)$ are bounded in each $B_R := \{(\phi, i) : \|\phi\| \leq R, i \leq R\}$, $R > 0$.

**Remark 3.1.** Recently, a functional Itô formula was developed in [11], which encouraged subsequent development (for example, [10, 41]). We briefly recall the main idea in what follows. Consider functions of the form

$$V(\phi, i) = f_1(\phi(0), i) + \int_{-r}^{0} g(t, i) f_2(\phi(t), i) dt,$$

where $f_2(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{Z}_+ \mapsto \mathbb{R}$ is a continuous function and $f_1(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{Z}_+ \mapsto \mathbb{R}$ is a function that is twice continuously differentiable in the first variable and $g(\cdot, \cdot) : \mathbb{R}_+ \times \mathbb{Z}_+ \mapsto \mathbb{R}$ be a continuously differentiable function in the first variable. Then at $(\phi, i) \in \mathcal{C} \times \mathbb{Z}_+$ we have (see [41] for the detailed computations)

$$V_t(\phi, i) = g(0, i)f_2(\phi(0), i) - g(-r, i)f_2(\phi(-r), i) - \int_{-r}^{0} f_2(\phi(t), i)dg(t, i),$$

$$\partial_k V(\phi, i) = \frac{\partial f_1}{\partial x_k}(\phi(0), i), \qquad \partial_{kl} V(\phi, i) = \frac{\partial^2 f_1}{\partial x_k \partial x_l}(\phi(0), i).$$

Let $V(\cdot, \cdot) \in \mathbb{F}$, we define the operator

$$\begin{aligned}\mathcal{L}V(\phi, i) =& V_t(\phi, i) + V_x(\phi, i)b(\phi(0), i) + \frac{1}{2}\operatorname{tr}\Big(V_{xx}(\phi, i)A(\phi(0), i)\Big) \\ &+ \sum_{j=1, j\neq i}^{\infty} q_{ij}(\phi)\big[V(\phi, j) - V(\phi, i\big] \\ =& V_t(\phi, i) + \sum_{k=1}^{n} b_k(\phi(0), i)V_k(\phi, i) + \frac{1}{2}\sum_{k,l=1}^{n} a_{kl}(\phi(0), i)V_{kl}(\phi, i) \\ &+ \sum_{j=1, j\neq i}^{\infty} q_{ij}(\phi)\big[V(\phi, j) - V(\phi, i)\big],\end{aligned} \tag{3.3}$$

for any bounded stopping time $\tau_1 \leq \tau_2$, we have the functional Itô formula:

$$\mathbb{E}V(X_{\tau_2}, \alpha(\tau_2)) = \mathbb{E}V(X_{\tau_1}, \alpha(\tau_1)) + \mathbb{E}\int_{\tau_1}^{\tau_2} \mathcal{L}V(X_s, \alpha(s))ds \tag{3.4}$$

if the expectations involved exist. Equation (3.4) is obtained by applying the functional Itô formula for general semimartingales given in [9, 10] specialized to our processes.

## 3.2 Recurrence and Ergodicity

First, we need some conditions for irreducibility of the process $\{(X_t, \alpha(t)) : t \geq 0\}$.

(H1) (a) For any $i \in \mathbb{Z}_+$, $A(x, i)$ is elliptic uniformly on each compact set, that is, for any $R > 0$, there is a $\theta_{R,i} > 0$ such that

$$y^\top A(x, i)y \geq \theta_{R,i}|y|^2 \quad \forall |x| \leq R,\ y \in \mathbb{R}^d. \tag{3.5}$$

(b) There is an $i^*$ satisfying that for any $i \in \mathbb{Z}_+$, there exist $i_1, \dots, i_k \in \mathbb{Z}_+$ and $\phi_1, \dots, \phi_{k+1} \in \mathcal{C}$ such that $q_{ii_1}(\phi_1) > 0$, $q_{i_l, i_{l+1}}(\phi_{l+1}) > 0, l = 1, \dots, k-1$, and $q_{i_k, i^*}(\phi_{k+1}) > 0$.

(H2) There exists an $i^* \in \mathbb{Z}_+$ such that

(a) $A(x, i^*)$ is elliptic uniformly on each compact set;

(b) for any $(\phi, i) \in \mathcal{C} \times \mathbb{Z}_+$, there exist positive integers $i = i_1, \dots, i_k = i^*$ satisfying $q_{i_l, i_{l+1}}(\phi) > 0, l = 1, \dots, k-1$.

Let $(X^{\phi,i}, \alpha^{\phi,i}(t))$ be the solution to (1.7) with initial data $(\phi, i) \in \mathcal{C} \times \mathbb{Z}_+$. To simplify the notation, we denote by $\mathbb{P}_{\phi,i}$ the probability measure conditioned on the initial data $(\phi, i)$, that is, for any $t > 0$,

$$\mathbb{P}_{\phi,i}\{(X_t, \alpha(t)) \in \cdot\} = \mathbb{P}\{(X_t^{\phi,i}, \alpha^{\phi,i}(t)) \in \cdot\},$$

and $\mathbb{E}_{\phi,i}$ the expectation associated with $\mathbb{P}_{\phi,i}$. To proceed, we state some auxiliary lemmas.

**Lemma 3.2.** *Let $\phi \in \mathcal{C}$ and $q_{ij}(\phi) > 0$. For any $\varepsilon > 0$, there is a $\delta > 0$ such that*

$$\inf_{\psi \in \mathcal{C}: \|\psi - \phi\| < \delta} \mathbb{P}_{\psi,i}\{\|X_\delta - \phi\| < \varepsilon, \alpha(\delta) = j\} > 0.$$

**Lemma 3.3.** *For any $i > 0$, $R > 0$ and $\varepsilon > 0$, there is a compact set $\mathcal{A} \in \mathcal{C}$ such that*

$$\inf_{\|\phi\| \le R} \mathbb{P}_{\phi,i}\{X_r \in \mathcal{A}, \alpha(r) = i\} > 0. \tag{3.6}$$

*Moreover, if (3.5) holds for $i$, then for any $k > 0$, there is a $T = T(k, i, R) > 0$ such that*

$$\inf_{\|\phi\| \le R} \mathbb{P}_{\phi,i}\{\|X_t\| > k \text{ for some } t \in [0, T]\} > 0. \tag{3.7}$$

**Lemma 3.4.** *Assume that either* (H1) *or* (H2) *is satisfied. There is a nontrivial measure* $\boldsymbol{\nu}(\cdot)$ *on* $\mathfrak{B}(\mathcal{C})$ *such that* $\boldsymbol{\nu}(\mathcal{D}) > 0$ *if* $\mathcal{D}$ *is a nonempty open subset of* $\mathcal{C}$ *and that for any* $R > 0$, $T > r$, *there is a* $d_{R,T} > 0$ *satisfying*

$$\mathbb{P}_{\phi,i^*}\{X_T \in \mathcal{B} \text{ and } \alpha(T) = i^*\} \geq d_{R,T}\boldsymbol{\nu}(\mathcal{B}),\, \mathcal{B} \in \mathfrak{B}(\mathcal{C}) \text{ given that } \|\phi\| \leq R. \tag{3.8}$$

The three lemmas above will be proved in the appendix.

**Lemma 3.5.** *Assume that either* (H1) *or* (H2) *holds. For any* $i \in \mathbb{Z}_+$, *there is a* $T_i > 0$ *such that for any* $T > T_i$ *and any open set* $\mathcal{B} \subset \mathcal{C}$, *we have*

$$\mathbb{P}_{\phi,i}\{X_T \in \mathcal{B}, \alpha(T) = i^*\} > 0,\, \phi \in \mathcal{C}$$

*where* $i^*$ *is as in* (H1) *or* (H2) *accordingly.*

*Proof.* Suppose that (H1) holds with $i = i_1, \ldots, i_k = i^* \in \mathbb{Z}_+$ and $\phi_1, \ldots, \phi_{k+1} \in \mathcal{C}$ such that $q_{i_l,i_{l+1}}(\phi_{l+1}) > 0, l = 1, \ldots, k-1$. Since $q_{i_l,i_{1+1}}(\phi_{l+1}) > 0$, it follows from Lemma 3.2 that

$$\mathbb{P}_{\psi,i_l}\{\|X_{\varepsilon_l} - \phi_{l+1}\| < 1, \alpha(\varepsilon_l) = i_{l+1}\} > 0 \text{ if } \|\psi - \phi_{l+1}\| < \varepsilon_l \tag{3.9}$$

for some $\varepsilon_l \in (0,1)$. In view of Lemma 3.4,

$$\mathbb{P}_{\psi,i_l}\{\|X_{1+r} - \phi_{l+1}\| < \varepsilon_l, \alpha(1+r) = i_l\} > 0 \text{ for any } \psi \in \mathcal{C}, \tag{3.10}$$

and

$$\mathbb{P}_{\psi,i^*}\{X_{1+r+T'} \in \mathcal{B}, \alpha(1+r+T') = i^*\} > 0 \text{ for any } \psi \in \mathcal{C}, T' \geq 0. \tag{3.11}$$

By (3.9), (3.10), and the Markov property of $(X_t, \alpha(t))$, we have

$$\mathbb{P}_{\psi,i_l}\{\|X_{1+r+\varepsilon_l} - \phi_l\| < 1, \alpha(1+r+\varepsilon_l) = i_{l+1}\} > 0 \text{ for any } \psi \in \mathcal{C}. \tag{3.12}$$

Using (3.11), (3.12), and applying the Kolmogorov-Chapman equation again, we obtain

$$\mathbb{P}_{\psi,i}\left\{X_{k(1+r)+\sum\varepsilon_l+T'} \in \mathcal{B}, \alpha\left(k(1+r)+\sum\varepsilon_l+T'\right)=i^*\right\} > 0 \text{ for any } \psi \in \mathcal{C}. \tag{3.13}$$

The lemma is proved with $T_i = k(2+r) \geq k(1+r) + \sum \varepsilon_l$.

Now, suppose that (H2) holds, it follows from Lemma 3.2 and the Kolmogorov-Chapman equation that

$$\mathbb{P}_{\phi,i}\{\|X_\varepsilon - \phi\| < 1, \alpha(\varepsilon) = i^*\} > 0 \tag{3.14}$$

for sufficiently small $\varepsilon$. The desired result follows from (3.11) and (3.14) with $T_i = 2 + r$. $\square$

**Lemma 3.6.** *Assume that either* (H1) *or* (H2) *holds. Let*

$$\eta_k = \inf\{t > 0 : \|X_t\| \vee \alpha(t) > k\}. \tag{3.15}$$

*Then for any* $(\phi, i) \in \mathcal{C} \times \mathbb{Z}_+$*, we have* $\mathbb{P}_{\phi,i}\{\eta_k < \infty\} = 1$*,* $\forall k \in \mathbb{Z}_+$*.*

*Proof.* Suppose that $p_0 = \mathbb{P}\{\eta_k < \infty\} < 1$. Since $A(x, i^*)$ is elliptic, in view of (3.7), there is a $T > 0$ such that

$$\mathbb{P}_{\varphi',i^*}\{\eta_k < T\} > 0,\ \forall k > 1, \|\varphi'\| \leq 1. \tag{3.16}$$

In view of Lemma 3.5 and (3.16), $\forall(\psi, j) \in \mathcal{C} \times \mathbb{Z}_+$ there are $T_{\psi,j}, p_{\psi,j} > 0$ such that

$$\mathbb{P}_{\psi,j}\{\eta_k < T_{\psi,j}\} > 2p_{\psi,j}. \tag{3.17}$$

Due to the Feller property of $(X_t, \alpha(t))$, there exists a $\delta_{\psi,j} > 0$ such that

$$\mathbb{P}_{\psi',i}\{\eta_k \leq T_{\phi,i^*}\} > p_{\psi,j},\ \forall\ \psi' \in \mathcal{C}, \|\psi - \psi'\| < \delta_{\psi,j}. \tag{3.18}$$

Since $\sigma(\cdot, i)$ and $b(\cdot, i)$ are locally compact for each $i \in \mathbb{Z}_+$, similar to Lemma 2.13, we can show that there is an $h_k > 0$ such that for any $t > 0$,

$$\mathbb{P}_{\phi,i}\left\{\frac{|X(s)-X(s')|}{(s-s')^{0.25}} \leq h_k,\ \forall\, 0 \vee (\eta_k \wedge t - r) \leq s' < s < \eta_k \wedge t\right\} > \frac{1+p_0}{2}. \tag{3.19}$$

Since the set

$$\mathcal{A}_k = \left\{\psi \in \mathcal{C} : |\psi| \leq k, \frac{|\psi(s)-\psi(s')|}{(s-s')^{0.25}} \leq h_k, \forall -r \leq s' < s \leq 0\right\}$$

is compact in $\mathcal{C}$, we have from (3.18) that there exist $T_k$ and $\tilde{p}_k$ such that

$$\mathbb{P}_{\psi,j}\{\eta_n < T_k\} > \tilde{p}_k > 0,\ \forall\, \psi \in \mathcal{A}_k, j \leq k. \tag{3.20}$$

Since $\lim_{t\to\infty} \mathbb{P}_{\phi,i}\{\eta_k < t\} = p_0 < 1$, there is a $T' > 0$ such that

$$p_0 \geq \mathbb{P}_{\phi,i}\{\eta_k \leq T'\} \geq p_0 - \frac{1-p_0}{2}\tilde{p}_k. \tag{3.21}$$

In view of (3.19) and (3.21), we have $\mathbb{P}_{\phi,i}\{T' < \eta_k, X_{T'} \in \mathcal{A}_k\} > \frac{1-p_0}{2}$. By the Markov property and (3.20),

$$\begin{aligned}\mathbb{P}_{\phi,i}\{T' < \eta_k < \infty\} &\geq \mathbb{P}_{\phi,i}\{X_{T'} \in \mathcal{A}_k, T' < \eta_k\} \\ &\geq \mathbb{E}_{\phi,i}\left[\mathbf{1}_{\{T'<\eta_k, X_{T'}\in\mathcal{A}_k\}}\mathbb{P}_{X_{T'},\alpha(T')}\{\eta_k < \infty\}\right] \\ &> \frac{1-p_0}{2}\tilde{p}_k.\end{aligned} \tag{3.22}$$

We have from (3.21) and (3.22) that

$$\begin{aligned}p_0 = \mathbb{P}_{\phi,i}\{\eta_k < \infty\} =&\mathbb{P}_{\phi,i}\{\eta_k \leq T'\} + \mathbb{P}_{\phi,i}\{T' < \eta_k < \infty\} \\ >&p_0 - \frac{1-p_0}{2}\tilde{p}_k + \frac{1-p_0}{2}\tilde{p}_k = p_0,\end{aligned}$$

which is a contradiction. Thus $p_0 = 1$. □

**Definition 3.7.** The process $\{(X_t, \alpha(t)) : t \geq 0\}$ is said to be recurrent (resp., positive recurrent) relative to a measurable set $\mathcal{E} \in \mathcal{C} \times \mathbb{Z}_+$ if

$$\mathbb{P}_{\phi,i}\{(X_t, \alpha(t)) \in \mathcal{E} \text{ for some } t \geq 0\} = 1$$

$$\big(\text{resp. } \mathbb{E}_{\phi,i}\left[\inf\{t > 0 : (X_t, \alpha(t)) \in \mathcal{E}\}\right] < \infty\big)$$

for any $(\phi, i) \in \mathcal{C} \times \mathbb{Z}_+$.

**Theorem 3.8.** *Suppose that either hypothesis* (H1) *or* (H2) *holds. Let $\mathcal{D}$ be a bounded open subset of $\mathcal{C}$ and $N$ be a finite subset of $\mathbb{Z}_+$. If $(X_t, \alpha(t))$ is recurrent relative to $\mathcal{D} \times N$ then $(X_t, \alpha(t))$ is recurrent relative to $\mathcal{D}' \times N'$ for any open set $\mathcal{D}' \subset \mathcal{C}$ and a finite set $N' \subset \mathbb{Z}_+$ containing $i^*$ with $i^*$ given in either* (H1) *or* (H2) *according to which hypothesis is satisfied.*

*Proof.* Let $(\phi_0, i_0) \in \mathcal{C} \times \mathbb{Z}_+$. In view of Lemma 3.3, there exists a compact set $\mathcal{A}_\mathcal{D} \subset \mathcal{C}$ such that

$$\inf_{\{(\psi,j) \in \mathcal{D} \times N\}} \mathbb{P}_{\psi,j}\{X_r \in \mathcal{A}_\mathcal{D}, \alpha(r) \in N\} := p_{\mathcal{D},N} > 0. \tag{3.23}$$

Since $i^* \in N'$, by Lemma 3.5, the Feller property of $(X_t, \alpha(t))$ and the compactness of $\mathcal{A}_\mathcal{D}$, there is a $T > 0$ such that

$$\inf_{\{(\psi,j) \in \mathcal{A}_\mathcal{D} \times N\}} \mathbb{P}_{\psi,j}\{X_T \in \mathcal{D}' \times N'\} \geq \varepsilon_0. \tag{3.24}$$

Define the stopping times $\vartheta_0 = 0, \vartheta_{k+1} = \inf\{t > \vartheta_k + T : X_{\eta_{k+1}} \in \mathcal{D} \times N\}$. By the hypothesis of the theorem,

$$\mathbb{P}_{\phi_0,i_0}\{\vartheta_k < \infty\} = 1, \ \forall\, k \in \mathbb{Z}_+.$$

On the other hand, it follows from (3.23) and (3.24) that

$$\mathbb{P}_{\psi,j}\{(X_T, \alpha(T) \in \mathcal{D}' \times N'\} \geq p_{\mathcal{D},N}\varepsilon_0, \text{ for all } (\psi, j) \in \mathcal{D} \times N. \tag{3.25}$$

Consider the events

$$A^k = \{(X_{\vartheta_k+T} \notin \mathcal{D}' \times N'\},\ k \in \mathbb{Z}_+.$$

By the strong Markov property of $(X_t, \alpha(t))$, we have

$$\mathbb{P}_{\phi_0,i_0}\left(\bigcap_{k'=k}^{\infty} A^{k'}\right) = \lim_{l\to\infty} \mathbb{P}_{\phi_0,i_0}\left(\bigcap_{k'=k}^{l} A^{k'}\right) \leq \lim_{l\to\infty}(1 - p_{\mathcal{D},N}\varepsilon_0)^{l-k} = 0.$$

Thus

$$\mathbb{P}_{\phi_0,i_0}\left(\bigcap_{k'=k}^{\infty} A^{k'}\right) = 0.$$

It indicates that the event $\left\{\left(X_{\vartheta_k+T}, \alpha(\vartheta_k + T)\right) \in \mathcal{D}' \times N\right\}$ must occur with probability 1.

□

**Theorem 3.9.** *Suppose that either hypothesis* (H1) *or* (H2) *holds. Let* $V(\cdot,\cdot) \in \mathbb{F}$ *such that*

$$\lim_{n\to\infty} \inf\{V(\phi, i) : |\phi(0)| \vee i \geq n\} = \infty.$$

*Suppose further that there are positive constants* $C$ *and* $H$ *such that*

$$\mathcal{L}V(\phi, i) \leq C\mathbf{1}_{\{V(\phi,i)\leq H\}}. \tag{3.26}$$

*Then the process* $(X_t, \alpha(t))$ *is recurrent relative to* $\mathcal{D} \times N$*, where* $\mathcal{D}$ *is any open bounded subset of* $\mathcal{C}$ *and* $N \subset \mathbb{Z}_+$ *contains* $i^*$.

*Proof.* Let $\upsilon_H = \inf\{t > 0 : V(X_t, \alpha(t)) \leq H\}$ and $\eta_k$ be defined as in Lemma 3.6. In view of Lemma 3.6, $\mathbb{P}_{\psi,j}\{\eta_k < \infty\} = 1, \forall k \in \mathbb{Z}_+$. Let $t > 0$. By Itô's formula

$$\mathbb{E}_{\psi,j} V\Big(X_{t\wedge \upsilon_H \wedge \eta_k}, \alpha(t \wedge \upsilon_H \wedge \eta_k)\Big) \le V(\psi, j).$$

Letting $t \to \infty$, we obtain

$$\mathbb{E}_{\psi,j} V\big(X_{\upsilon_H \wedge \eta_k}, \alpha(\upsilon_H \wedge \eta_k)\big) \le V(\psi, j),$$

which implies

$$\mathbb{P}_{\psi,j}\{\upsilon_H > \eta_k\} \le \frac{V(\psi, j)}{\inf\{V(\phi, i) : |\phi(0)| \vee i \ge k\}}.$$

Letting $k \to \infty$ yields $\mathbb{P}_{\psi,j}\{\upsilon_H > \eta_k\} \to 0$. Thus,

$$\mathbb{P}_{\psi,j}\{\upsilon_H < \infty\} = 1, \ \forall (\psi, j) \in \mathcal{C} \times \mathbb{Z}_+. \tag{3.27}$$

Now, let $k_0 > 0$ such that $\inf\{V(\phi, i) : |\phi(0)| \vee i \ge k_0\} \ge 2(H+C+r)$. For any $(\psi, j) \in \mathcal{C} \times \mathbb{Z}_+$ satisfying $V(\psi, j) \le H$. We have from (3.26) and Itô's formula that

$$\mathbb{E}_{\psi,j} V\big(X_{r\wedge \eta_{n_0}}, \alpha(r \wedge \eta_{n_0})\big) \le H + C + r,$$

which implies

$$\mathbb{P}_{\psi,j}\{\eta_{n_0} < r\} \le \frac{H + C + r}{2(H + C + r)} \le \frac{1}{2}. \tag{3.28}$$

Thus,

$$\mathbb{P}_{\psi,j}\{\|X_r\| < n_0, \alpha(r) < n_0\} \ge \mathbb{P}_{\psi,j}\{\eta_{n_0} > r\} > \frac{1}{2} \text{ provided } V(\psi, j) \le H. \tag{3.29}$$

Now, fix $(\phi_0, i_0) \in \mathcal{C} \times \mathbb{Z}_+$. By (3.27) and Lemma 3.6, we can define almost surely finite stopping times

$$\begin{aligned}\zeta_1 &= \inf\{t \geq 0 : \ V(X_t, \alpha(t)) \leq H\},\\ \zeta_{2k} &= \inf\{t \geq \zeta_{2k-1} + r : \ |X_t| \vee \alpha(t) \geq n_0\},\\ \zeta_{2k+1} &= \inf\{t \geq \zeta_{2k} : \ V(X_t, \alpha(t)) \leq H\}.\end{aligned} \tag{3.30}$$

Define events $B_k = \{\|X_{\zeta_{2k+1}+r}\| \vee \alpha(\zeta_{2k+1}+r) \leq n_0\}$. In view of (3.29) and the strong Markov property of $(X_t, \alpha(t))$, we can use standard arguments in Theorem 3.8 to show that

$$\mathbb{P}_{\phi_0, i_0}\{B_k \text{ occurs for some } k\} = 1.$$

Thus, $(X_t, \alpha(t))$ is recurrent relative to $\{(\phi, i) : \|\phi\| \vee i \leq n_0\}$. Combining this with Theorem 3.8 yields the desired result. □

**Example 3.10.** Let

$$q_{12}(\phi) = 1, q_{1j}(\phi) = 0 \ \text{ for } j \geq 3;$$

$$q_{i,i-1}(\phi) = C_i + (1 + \|\phi\|)^{-1}, q_{i,i+1}(\phi) = C_i + (1 + \|\phi\|)^{-1} \ \text{ for } i \geq 2, \ C_i \geq 0;$$

$$q_{ij}(\phi) = 0 \ \text{ for } i \geq 2, j \notin \{i-1, i, i+1\}.$$

Suppose the switching diffusion is given by

$$dX(t) = \sigma(X(t), \alpha(t))dW(t) - X(t)b(X(t), \alpha(t))dt$$

where $b(x, i) > 0, \sigma(x, i)$ are locally Lipchitz in $x$ and uniformly bounded in $K \times \mathbb{Z}_+$ for each compact set $K \in \mathbb{R}$. Let $f(x)$ be twice continuously differentiable such that $f(x) > 0$ and $f(x) = |x|$ if $|x| \geq 1$. Let

$$\kappa := \sup_{|x| \leq 1, i \in \mathbb{Z}_+} \left| -\left[\frac{df}{dx}(x)\right] x b(x, i) + \frac{1}{2}\left[\frac{d^2 f}{dx^2} f(x)\right] \sigma^2(x, i)\right| < \infty. \tag{3.31}$$

Let

$$V(\phi, i) = f(\phi(0)) + 2\kappa i.$$

Direct computation leads to

$$\mathcal{L}V(\phi, i) = \begin{cases} -\left[\dfrac{df}{dx}(\phi(0))\right]\phi(0)b(\phi(0), i) + \frac{\sigma^2(i)}{2}\left[\frac{d^2 f}{dx^2}(\phi(0))\right] - 2\kappa(1 + \|\phi\|)^{-1} & \text{if } i > 1, \\ -\left[\dfrac{df}{dx}(\phi(0))\right]\phi(0)b(\phi(0), i) + \left[\frac{d^2 f}{dx^2}(\phi(0))\right] + 2\kappa & \text{if } i = 1. \end{cases} \tag{3.32}$$

In view of (3.31) and the fact that $\frac{d}{dx}f(x) = \operatorname{sgn}(x)$, $\frac{d^2}{dx^2}f(x) = 0$ for $|x| \geq 1$, $i > 1$ we have

$$\mathcal{L}V(\phi, i) \leq 0 \,\forall \phi \in \mathcal{C}, i > 1. \tag{3.33}$$

By (3.32), if we assume further $\lim_{x\to\infty} |x|b(x, 1) = \infty$, then we can verify that

$$\mathcal{L}V(\phi, 1) \leq \tilde{C}_1 \mathbf{1}_{\{|\phi(0)| < \tilde{H}\}} - \tilde{C}_2, \forall \phi \in \mathcal{C}, \tag{3.34}$$

where $\tilde{C}_1, \tilde{C}_2, \tilde{H}$ are some positive constants. In view of (3.33) and (3.34), we can easily check that (3.26) holds in this example, for $V(\phi, i)$ defined above and suitable $C, H$. Thus, if there exists $i^* \in \mathbb{Z}_+$ such that $\sigma(x, i^*) \neq 0$ for any $x \in \mathbb{R}$, then the conclusion of Theorem 3.9 holds for this example.

To proceed, let us recall some technical concepts and results needed to prove the main theorem. Let $\mathbf{\Phi} = (\Phi_0, \Phi_1, \ldots)$ be a discrete-time Markov chain on a general state space $(E, \mathfrak{E})$, where $\mathfrak{E}$ is a countably generated $\sigma$-algebra. Denote by $\mathcal{P}$ the Markov transition kernel for $\mathbf{\Phi}$. If there is a non-trivial $\sigma$-finite positive measure $\varphi$ on $(E, \mathfrak{E})$ such that for any

$A \in \mathfrak{E}$ satisfying $\varphi(A) > 0$ we have

$$\sum_{n=1}^{\infty} \mathcal{P}^n(x, A) > 0, \; x \in E$$

where $\mathcal{P}^n$ is the $n$-step transition kernel of $\mathbf{\Phi}$ then the Markov chain $\mathbf{\Phi}$ is called *$\varphi$-irreducible.* It can be shown (see [39]) that if $\mathbf{\Phi}$ is $\varphi$-irreducible, then there exists a positive integer $d$ and disjoint subsets $E_0, \ldots, E_{d-1}$ such that for all $i = 0, \ldots, d-1$ and all $x \in E_i$ we have

$$\mathcal{P}(x, E_j) = 1 \text{ where } j = i + 1 \pmod{d}$$

and

$$\varphi\left(E \setminus \bigcup_{i=0}^{d-1} E_i\right) = 0.$$

The smallest positive integer $d$ satisfying the above is called the period of $\mathbf{\Phi}$. An *aperiodic* Markov chain is a chain with period $d = 1$. A set $C \in \mathfrak{E}$ is called *petite* if there exists a non-negative sequence $(a_n)_{n \in \mathbb{Z}_+}$ with $\sum_{n=1}^{\infty} a_n = 1$ and a nontrivial positive measure $\nu$ on $(E, \mathfrak{E})$ satisfying that

$$\sum_{n=1}^{\infty} a_n \mathcal{P}^n(x, A) \geq \nu(A), \;\; x \in C, A \in \mathfrak{E}.$$

**Lemma 3.11.** *Assume either* (H1) *or* (H2) *holds. The Markov chain* $\{(X_k, \alpha(k)) : k \in \mathbb{Z}_+\}$ *is irreducible and aperiodic. Moreover, for every bounded set* $\mathcal{D} \in \mathcal{C}$ *and a finite set* $N \in \mathbb{Z}_+$*, the set* $\mathcal{D} \times N$ *is petite for* $\{(X_k, \alpha(k)) : k \in \mathbb{Z}_+\}$.

*Proof.* Similar to (3.25), there are $k_0 \in \mathbb{Z}_+, k_0 > r$, $\tilde{d}_{\mathcal{D},N} > 0$ such that

$$\mathbb{P}_{\phi,i}\{X_t \in \mathcal{D}, \alpha_{k_0} = i^*\} \geq \tilde{d}_{D,N} \;\text{ for all }\; (\phi, i) \in \mathcal{D} \times N. \tag{3.35}$$

By the Markov property, we deduce from (3.8) and (3.35) that for any $k > r$, there exists a

$\widehat{d}_{\mathcal{D},N,k} > 0$ such that

$$\mathbb{P}_{\phi,i}\{X_{k+k_0} \in \mathcal{B} \text{ and } \alpha(k+k_0)i^*\} \geq \widehat{d}_{\mathcal{D},N,k}\boldsymbol{\nu}(\mathcal{B}),\ \mathcal{B} \in \mathfrak{B}(\mathcal{C}),\ (\phi,i) \in \mathcal{D} \times N. \tag{3.36}$$

Let $\widehat{\boldsymbol{\nu}}(\cdot)$ be the measure on $\mathfrak{B}(\mathcal{C} \times \mathbb{Z}_+)$ given by $\widehat{\boldsymbol{\nu}}(\mathcal{E}) = \boldsymbol{\nu}(\{\phi \in \mathcal{C} : (\phi, i^*) \in \mathcal{E}\})$ for $\mathcal{E} \in \mathfrak{B}(\mathcal{C} \times \mathbb{Z}_+)$. Then (3.36) can be rewritten as

$$\mathbb{P}_{\phi,i}\{(X_{k+k_0}, \alpha(k+k_0)) \in \mathcal{E}\} \geq \widehat{d}_{D,N,k}\widehat{\boldsymbol{\nu}}(\mathcal{E}),\ \mathcal{E} \in \mathfrak{B}(\mathcal{C} \times \mathbb{Z}_+),\ (\phi,i) \in \mathcal{D} \times N. \tag{3.37}$$

It can be checked that (3.37) implies that the Markov chain $\{(X_k, \alpha(k)) : k \in \mathbb{Z}_+\}$ is $\widehat{\boldsymbol{\nu}}$-irreducible and every nonempty bounded set in $\mathcal{C} \times \mathbb{Z}_+$ is petite. Moreover, suppose that $(X_k, \alpha(k))$ is not aperiodic. Then, there are disjoint set $\mathcal{E}_0, \ldots, \mathcal{E}_{d-1}, d > 1$ such that

$$\widehat{\boldsymbol{\nu}}\left((\mathcal{C} \times \mathbb{Z}_+) \setminus \bigcup_{j=0}^{d-1} \mathcal{E}_j\right) = 0 \tag{3.38}$$

and

$$\mathbb{P}_{\phi,i}\{(X_1, \alpha(1)) \in \mathcal{E}_j\} = 1 \text{ if } j = j' + 1 \pmod d \text{ if } (\phi, i) \in \mathcal{E}_{j'},$$

which results in

$$\mathbb{P}_{\phi,i}((X_m, \alpha(m)) \in \mathcal{E}_j) = \begin{cases} 1 & \text{where } m = j+1 \pmod d \\ 0 & \text{otherwise} \end{cases} \quad \text{if } (\phi, i) \in \mathcal{E}_j. \tag{3.39}$$

In view of (3.36), for any $m > k_0 + r$ and $(\phi, i) \in \mathcal{C} \times \mathbb{Z}_+$, there is a $\tilde{p}_{\phi,i,k} > 0$ such that

$$\mathbb{P}_{\phi,i}\{(X_m, \alpha(m)) \in \mathcal{E}\} \geq \tilde{p}_{\phi,i,k}\widehat{\boldsymbol{\nu}}(\mathcal{E}) \tag{3.40}$$

for any measurable set $\mathcal{E} \in \mathfrak{B}(\mathcal{C} \times \mathbb{Z}_+)$. As a result of (3.39) and (3.40), we have that

$\widehat{\nu}(\mathcal{E}_j) = 0$ for any $j = 0, \ldots, d-1$. Thus,

$$\widehat{\nu}\left((\mathcal{C} \times \mathbb{Z}_+) \setminus \bigcup_{j=0}^{d-1} \mathcal{E}_j\right) = \widehat{\nu}(\mathcal{C} \times \mathbb{Z}_+) > 0, \tag{3.41}$$

which contradicts (3.38). This contradiction implies that $(X_k, \alpha(k))$ is aperiodic. □

**Theorem 3.12.** *Suppose that either* (H1) *or* (H2) *holds. Let* $V(\cdot,\cdot) \in \mathbb{F}$ *such that*

$$\lim_{n\to\infty} \inf\{V(\phi, i) : |\phi(0)| \vee i \geq n\} = \infty. \tag{3.42}$$

*Suppose further that there are positive constants* $C_1, C_2$ *and* $H$ *such that*

$$\mathcal{L}V(\phi, i) \leq -C_1 + C_2 \mathbf{1}_{\{V(\phi,i)\geq H\}}. \tag{3.43}$$

*Then,* $(X_t, \alpha(t))$ *is positive recurrent relative to any set of the form* $\mathcal{D} \times N$ *where* $\mathcal{D}$ *is a nonempty open set of* $\mathcal{C}$ *and* $N \ni i^*$ *with* $i^*$ *given in either* (H1) *and* (H2). *Moreover, there is a unique invariant probability measure* $\mu^*$*, and for any* $(\phi, i) \in \mathcal{C} \times \mathbb{Z}_+$

$$\lim_{t\to\infty} \|P(t, (\phi, i), \cdot) - \mu^*\|_{TV} = 0.$$

*Proof.* Let $\upsilon_H = \inf\{t \geq 0 : V(X_t, \alpha(t)) \leq H\}$. In view of the functional Itô formula,

$$\begin{aligned} \mathbb{E}_{\phi,i} V(X_{1\wedge\upsilon_H}, \alpha(1 \wedge \upsilon_H)) =& V(\phi, i) + \mathbb{E}_{\phi,i} \int_0^{1\wedge\upsilon_H} \mathcal{L}V(X_s, \alpha(s))ds \\ \leq& V(\phi, i) - C_1 \mathbb{E}_{\phi,i} 1 \wedge \upsilon_H \\ \leq& V(\phi, i) - C_1 \mathbb{P}_{\phi,i}\{\upsilon_H \geq 1\}. \end{aligned} \tag{3.44}$$

For any $t \leq 1$ and $V(\phi, i) \leq H$, we have

$$\begin{aligned}\mathbb{E}_{\phi,i} V(X_t, \alpha(t)) =& V(\phi, i) + \mathbb{E}_{\phi,i} \int_0^t \mathcal{L}V(X_s, \alpha(s))ds \\ \leq& V(\phi, i) + C_2 t \\ \leq& H + C_2.\end{aligned} \tag{3.45}$$

It follows from (3.45) and the strong Markov property of $(X_t, \alpha(t))$ that

$$\begin{aligned}\mathbb{E}_{\phi,i}\left[\mathbf{1}_{\{\upsilon_H<1\}} V(X_1, \alpha(1))\right] \leq& (H + C_2)\mathbb{P}_{\phi,i}\{\upsilon_H < 1\} \\ \leq& 2(H + C_2) - (H + C_2)\mathbb{P}_{\phi,i}\{\upsilon_H < 1\}.\end{aligned} \tag{3.46}$$

Let $\mathcal{C}_V := \{(\psi', j') : V(\psi, j) \leq 2(H + C_2)\}$. In view of (3.44) and (3.46),

$$\begin{aligned}\mathbb{E}_{\phi,i} V(X_1, \alpha(1)) \leq& \mathbb{E}_{\phi,i}\left[\mathbf{1}_{\{\upsilon_H<1\}} V(X_1, \alpha(1))\right] + \mathbb{E}_{\phi,i} V(X_{1\wedge\upsilon_H}, \alpha(1 \wedge \upsilon_H)) \\ \leq& V(\phi, i) - \min\{C_1, H + C_2\} + 2(H + C_2)\mathbf{1}_{\{(\phi,i)\in\mathcal{C}_V\}}.\end{aligned} \tag{3.47}$$

Let $n_0 \in \mathbb{Z}_+$ such that

$$V(\phi, i) \geq 2(2H + 2C_2 + C_2 r) \text{ for any } \|\phi\| \vee i \geq n_0, \tag{3.48}$$

and define $\widehat{\zeta}_V = \inf\{t \geq 0 : V(X_t, \alpha(t)) \geq 2(2H + 2C_2 + C_2 r)\}$. Similar to (3.28), we have

$$\mathbb{P}_{\phi,i}\{\widehat{\zeta}_V \leq r\} \leq \frac{1}{2} \text{ for } (\phi, i) \in \mathcal{C}_V. \tag{3.49}$$

Thus,

$$\begin{aligned}\mathbb{P}_{\phi,i}\{\|X_r\| \vee \alpha(r) \leq n_0\} \geq& \mathbb{P}_{\phi,i}\{V(X_r, \alpha(r)) \geq H + C_2 r + 1\} \\ \geq& 1 - \mathbb{P}_{\phi,i}\{\widehat{\zeta}_V \leq r\} = \frac{1}{2}, \ (\phi, i) \in \mathcal{C}_H.\end{aligned} \tag{3.50}$$

In view of (3.37) and (3.50),

$$\mathbb{P}_{\phi,i}\{(X_{k+k_0}, \alpha(k + k_0)) \in \mathcal{E}\} \geq \widehat{d}_{H,k}\widehat{\boldsymbol{\nu}}(\mathcal{E}),\ \mathcal{E} \in \mathfrak{B}(\mathcal{C} \times \mathbb{Z}_+), \text{ if } V(\phi, i) \leq H, k > r \tag{3.51}$$

for some $\widehat{d}_{H,k} > 0$. Thus, the set $\{(\phi, i) \in \mathcal{C} \times \mathbb{Z}_+ : V(\phi, i) \le H\}$ is petite for $\{(X_k, \alpha(k)) : k \in \mathbb{Z}_+\}$. Using this and (3.51), it follows from [51, Theorem 2.1] (or [33]) that

$$\lim_{n\to\infty} \|P(n, (\phi, i), \cdot) - \mu^*\|_{TV} = 0$$

where $P(t, (\phi, i), \cdot)$ is the transition probability of $(X_t, \alpha(t))$ and $\mu^*$ is an invariant probability measure of the Markov chain $\{X_k, \alpha(k), k \in \mathbb{Z}_+\}$. It is easy to show that $\mu^*$ is also an invariant probability measure of the process $\{(X_t, \alpha(t))\}$. Thus $\|P(t, (\phi, i), \cdot) - \mu^*\|_{TV}$ is decreasing in $t$, which leads to

$$\lim_{t\to\infty} \|P(t, (\phi, i), \cdot) - \mu^*\|_{TV} = 0.$$

Now we show that the process $(X_t, \alpha(t))$ is positive recurrent. Similar to (3.44), we deduce from the functional Itô formula that

$$\mathbb{E}_{\phi,i} \upsilon_H \le C_1^{-1} V(\phi, i).$$

Owing to this and (3.51), we can use the arguments in the proof of [57, Lemma 3.6] to show that $(X_t, \alpha(t))$ is positive recurrent. □

**Example 3.13.** In Example 3.10, if we assume further that

$$\lim_{|x|\to\infty} \inf_{i\in\mathbb{Z}_+} \{|x| b(x, i)\} > 0 \tag{3.52}$$

then it follows from (3.31) and (3.32) that

$$\mathcal{L}V(\phi, i) \le -\widehat{C} \ \text{ for } \phi \in \mathcal{C}, i \ge 2,$$

for some positive constant $\widehat{C}$. This combined with (3.34) shows that (3.43) holds for the switching diffusion $(X(t), \alpha(t))$ and the function $V(\phi, i)$ in Example 3.10. Thus the conclusion of Theorem 3.12 holds for the switching diffusion in Example 3.10 with the additional condition (3.52).

**Theorem 3.14.** *Suppose that either* (H1) *or* (H2) *holds. Let* $V(\cdot,\cdot) \in \mathbb{F}$ *such that*

$$\lim_{n\to\infty} \inf\{V(\phi, i) : |\phi(0)| \vee i \geq n\} = \infty. \tag{3.53}$$

*Suppose further that there are* $C_1$ *and* $C_2 > 0$ *such that*

$$\mathcal{L}V(\phi, i) \leq -C_1 V(\phi, i) + C_2. \tag{3.54}$$

*Then, there is a unique invariant probability measure* $\mu^*$ *and* $\theta > 0$ *such that for any* $(\phi, i) \in \mathcal{C} \times \mathbb{Z}_+$

$$\lim_{t\to\infty} \exp(\theta t)\|P(t, (\phi, i), \cdot) - \mu^*\|_{TV} = 0. \tag{3.55}$$

*Proof.*

$$\begin{aligned}
&\mathbb{E}_{\phi,i} \exp\{C_1(\eta_k \wedge t)\} V(X_{\eta_k\wedge t}, \alpha(\eta_k \wedge t)) \\
&= V(\phi, i) + \mathbb{E}_{\phi,i} \int_0^{\eta_k\wedge t} e^{C_1 s}[\mathcal{L}V(X_s, \alpha(s)) + C_1 V(X_s, \alpha(s))]ds \\
&\leq V(\phi, i) + C_2 \mathbb{E}_{\phi,i} \int_0^{\eta_k\wedge t} e^{C_1 s} ds \\
&\leq V(\phi, i) + C_1^{-1} C_2 e^{C_1 t}.
\end{aligned} \tag{3.56}$$

Letting $k \to \infty$, we obtain

$$\mathbb{E}_{\phi,i} V(X_t, \alpha(t)) \leq e^{-C_1 t} V(\phi, i) + C_1^{-1} C_2 \tag{3.57}$$

Let $\gamma_1 = e^{-C_1}$ and $\gamma_2 \in (\gamma_1, 1)$. It follows from (3.57) and (3.46) that

$$\begin{aligned}\mathbb{E}V(X_1, \alpha(1)) \leq & \gamma_1 V(\phi, i) + C_1^{-1} C_2 \\ = & \gamma_2 V(\phi, i) + \Big[C_1^{-1} C_2 - (\gamma_2 - \gamma_1) V(\phi, i)\Big] \\ \leq & \gamma_2 V(\phi, i) + [C_1^{-1} C_2] \mathbf{1}_{\{V(\phi,i) \leq H'\}}\end{aligned} \tag{3.58}$$

where $H' = C_1^{-1} C_2 (\gamma_2 - \gamma_1)^{-1}$. Similar to (3.51), the set $\{(\phi, i) \in \mathcal{C} \times \mathbb{Z}_+ : V(\phi, i) \leq H'\}$ is petite, which combined with (3.58) implies the existence of $\gamma_3 \in (0, 1)$ such that

$$\lim_{n\to\infty} \gamma_3^n \|P(n, (\phi, i), \cdot) - \mu^*\|_{TV} = 0$$

due to a well-known theorem (see, e.g., [33]). Then (3.55) follows from (3.58) and the decreasing property of $\|P(t, (\phi, i), \cdot) - \mu^*\|_{TV}$ in $t$. □

**Example 3.15.** Suppose that

$$\begin{aligned}&q_{12}(\phi) = 1, q_{1j}(\phi) = 0 \text{ for } j \geq 3; \\ &q_{i,1}(\phi) = 2\int_{-r}^{0} |\phi(s)| ds, q_{i,i+1}(\phi) = i\int_{-r}^{0} |\phi(s)| ds \text{ for } i \geq 2 \\ &q_{ij}(\phi) = 0 \text{ for } i \geq 2, j \notin \{1, i, i+1\}.\end{aligned}$$

and that the equation for the diffusion part is

$$dX(t) = \sigma(X(t), \alpha(t)) dW(t) - b(X(t), \alpha(t)) X(t) dt$$

where $\sigma(x, i), b(x, i)$ are locally Lipchitz in $x$ and uniformly bounded in $K \times \mathbb{Z}_+$ for each compact set $K \in \mathbb{R}$. Let $V(\phi, i)$ be defined as in Example (3.10). Similar to Examples 3.10 and 3.13, under the assumption that $b := \inf_{(x,i)\in\mathbb{R}\times\mathbb{Z}_+}\{b(x, i)\} > 0$. one can show that (3.54) holds in this example with this function $V$. Thus, the conclusion of Theorem 3.9 holds for this example. if there exists $i^* \in \mathbb{Z}_+$ such that $\sigma(x, i^*) \neq 0$ for any $x \in \mathbb{R}$.

**Example 3.16.** Let

$$q_{12}(\phi) = 1, q_{1j}(\phi) = 0 \text{ for } j \geq 3;$$

$$q_{i,i-1}(\phi) = C_i + 2|\phi(0)|, q_{i,i+1} = C_i + |\phi(-r)| \text{ for } i \geq 2,\ C_i \geq 0;$$

$$q_{ij}(\phi) = 0 \text{ for } i \geq 2, j \notin \{i-1, i, i+1\}.$$

Consider the general equation for diffusion (1.5), where $\sigma(x,i), b(x,i)$ are locally Lipchitz in $x$ in $K \times \mathbb{Z}_+$ for each compact set $K \in \mathbb{R}$. Suppose there is a function $U(x) : \mathbb{R}^n \mapsto \mathbb{R}_+$ satisfying

- $U(x)$ is twice continuously differentiable in $x$.
- $\lim_{|x|\to\infty} U(x,i) = \infty$.
- There are positive constants $C_1, C_2, H$ such that

$$\mathcal{L}_i U(x) \leq -C_1 U(x) + C_2. \tag{3.59}$$

Let $V(x,i) = U(x) + i + \int_0^t \exp\{\frac{\ln 2}{r}(s+r)\}ds$. By Remark 3.1,

$$\mathcal{L}V(\phi,i) = \begin{cases} \mathcal{L}_i U(x) - i - \dfrac{\ln 2}{r}\displaystyle\int_0^t \exp\left\{\dfrac{\ln 2}{r}(s+r)\right\} ds + 2 & \text{if } i > 1 \\ \mathcal{L}_i U(x) - \dfrac{\ln 2}{r}\displaystyle\int_0^t \exp\left\{\dfrac{\ln 2}{r}(s+r)\right\} ds + 1 & \text{if } i = 1 \end{cases} \tag{3.60}$$

As a consequence of (3.59) and (3.60), there are $C_3$ and $C_4 > 0$ such that

$$\mathcal{L}V(\phi,i) \leq -C_3 V(\phi,i) + C_4 \text{ for } (\phi,i) \in \mathcal{C} \times \mathbb{Z}_+.$$

Thus, the conclusion of Theorem 3.9 holds for this example if there exists $i^* \in \mathbb{Z}_+$ such that $A(x,i^*)$ is elliptic.

## 3.3 Recurrence of Past-Independent Switching Diffusions

This section is devoted mainly to characterizing the recurrence of $(X(t), \alpha(t))$ using the corresponding system of partial differential equations when the switching intensities of $\alpha(t)$ depends only on the current state of $X(t)$, that is $q_{ij}(\cdot), i, j \in \mathbb{Z}_+$ are functions on $\mathbb{R}^n$ rather than on $\mathcal{C}$. To simplify the presentation, throughout this section, we set $q_{ii}(x) = 0$ for $(x, i) \in \mathbb{R}^n \times \mathbb{Z}_+$. Thus, $q_i(x) = -\sum_{j\in\mathbb{Z}_+} q_{ij}(x)$. In this section, we use the following assumption.

**Assumption 3.1.** Suppose that

1. either Assumption 2.3 or Assumption 2.4 holds with $\phi \in \mathcal{C}$ replaced by $x \in \mathbb{R}^n$;
2. for each $i \in \mathbb{Z}_+$, $A(x, i)$ is uniformly elliptic in each compact set;
3. for any $x \in \mathbb{R}^n$, there are $\widehat{q} = \widehat{q}(x) > 0$ and $n_{\widehat{q}} = n_{\widehat{q}}(x) > 0$ such that

$$\sum_{j\le n_{\widehat{p}}} q_{ij}(x) \ge \widehat{q} \ \text{ for any } \ i > n_{\widehat{q}}. \tag{3.61}$$

**Remark 3.17.** We note the following facts.

- Part 3 of Assumption 3.1 stems from a familiar condition for uniform ergodicity of the Markov chain having a countable state space. In other word, if (3.61) holds, for each $x \in \mathbb{R}^n$, the Markov chain $\widehat{\alpha}^x(t)$ with generator $Q(x)$ has a property that

$$\sup_{i\in\mathbb{Z}_+} \mathbb{E}_i \varsigma < \infty$$

  where $\varsigma$ is the first time the process $\widehat{\alpha}^x(t)$ jumps to $\{1, \ldots, n_0\}$ and $\mathbb{E}_i$ is the expectation with condition $\widehat{\alpha}^x(0) = i$.
- Since $q_{ij}(x)$ is continuous in $x \in \mathbb{R}^n$, with the use of the Heine-Borel covering theorem, it is easy to show that for any bounded set $D \in \mathbb{R}^n$, there is $\varepsilon_0 = \varepsilon_0(D)$ and $n_0 = n_0(D)$

such that

$$\sum_{j\le n_0} q_{ij}(x) \ge \varepsilon_0 \ \text{ for any } \ i > n_0, x \in D. \tag{3.62}$$

For an open set $D \subset \mathbb{R}^n$, define

$$\widetilde{\tau}_D = \inf\{t \ge 0 : X(t) \notin D\},$$

and $W^{2,p}_{loc}(D)$ is the set of functions $u : \overline{D} \mapsto \mathbb{R}$ that has generalized derivatives $D^\beta u$ for any multiple-index $\beta = (\beta_1, \dots, \beta_n)$ with $|\beta| = \sum \beta_i \le 2$ satisfying $D^\beta u \in L^p_{loc}(D)$ if $|\beta| \le 2$. Let $\mathbb{H}^p(D)$ be the set of functions $u(x,i)$ in $\overline{\mathcal{D}} \times \mathbb{Z}_+$ satisfying that

- For each $i \in \mathbb{Z}_+$, $u(\cdot, i) \in W^{2,p}_{loc}(D)$ and $u(\cdot, i)$ is continuous in the closure $\overline{D}$ of $D$.
- For any compact set $K \subset \mathbb{R}^n$, $\sup_{(x,i)\in(K\cap\overline{D})\times\mathbb{Z}_+}\{u(x,i)\} < \infty$.

Let $\mathcal{L}_i$ be defined as in Remark 2.4. We state the two main results of this section.

**Theorem 3.18.** *Suppose that Assumption* 3.1 *holds. Let $D_1$ be a bounded open set of $\mathbb{R}^n$ with $\partial D_1 \in C^2$ and $D = \overline{D}_1^{\,c}$ be the complement of $\overline{D}_1$. Let $p > n$. The process $X(t)$ is recurrent relative to $D_1$, if and only if the Dirichlet problem*

$$\begin{cases} \mathcal{L}_i u(x,i) - q_i(x)u(x,i) + \sum_{j\in\mathbb{Z}_+} q_{ij}(x)u(x,j) = 0 \ \text{ in } D \times \mathbb{Z}_+ \\ u(x,i) = f(x,i) \ \text{ on } \partial D \times \mathbb{Z}_+. \end{cases} \tag{3.63}$$

*has a unique solution in $\mathbb{H}^p(D)$ given that $f(x,i)$ is continuous in $x \in \partial D$ and bounded in $\partial\mathcal{D} \times \mathbb{Z}_+$.*

**Theorem 3.19.** *Suppose that Assumption* 3.1 *holds. Let $D_1$ be a bounded open set of $\mathbb{R}^n$ with boundary $\partial D_1 \in C^2$ and $D = D_1^c$ be its complement. Let $p > n$. Suppose further that* *for each compact set $K \in \mathbb{R}^n$, the function $u(x,i) = \mathbb{E}_{x,i}\widetilde{\tau}_D$ is bounded in $K \times \mathbb{Z}_+$. Then $\{u(x,i)\} \in \mathbb{H}^p(D), p > n$ is a strong solution to*

$$\begin{cases} \mathcal{L}_i u(x,i) - q_i(x)u(x,i) + \sum_{j\in\mathbb{Z}_+} q_{ij}(x)u(x,j) = -1 \ \text{in } D \times \mathbb{Z}_+ \\ u(x,i) = 0 \ \text{on } \partial D \times \mathbb{Z}_+. \end{cases} \tag{3.64}$$

*The solution is unique in $\mathbb{H}^p(D), p > n$.*

**Lemma 3.20.** *Let $D \subset \mathbb{R}^n$ be an open bounded set with $\partial D \in C^2$. The Dirichlet problem*

$$\begin{cases} \mathcal{L}_i u(x,i) - q_i(x)u(x,i) + \sum_{j\in\mathbb{Z}_+} q_{ij}(x)u(x,j) = f(x,i) \ \text{in } D \times \mathbb{Z}_+ \\ u(x,i)\big|_{\partial D} = \phi(x,i) \ \text{on } \partial D \times \mathbb{Z}_+. \end{cases} \tag{3.65}$$

*has a unique strong solution $\{u(x,i)\} \in \mathbb{H}^p(D), p > n$ if $\phi(x,i)$ and $f(x,i)$ are continuous and bounded on $\partial D \times \mathbb{Z}_+$ and $D \times \mathbb{Z}_+$ respectively.*

*Proof.* The proof is motivated by that of [12, Proposition A]. However, because there are infinitely many equations, significant modification is needed. Let $p > n$ and

$$\widehat{M} = \sup_{(x,i)\in\partial D\times\mathbb{Z}_+} \{|\phi(x,i)|\} + \sup_{(x,i)\in D\times\mathbb{Z}_+} \{|f(x,i)|\} < \infty. \tag{3.66}$$

By [54, Theorem 9.1.5], for each $i \in \mathbb{Z}_+$, there exists a strong solution $u_0(x,i) \in W^{2,p}_{loc}(D) \cap C(\overline{D})$ to

$$\begin{cases} \mathcal{L}_i u_0(x,i) - q_i(x)u(x,i) = 0 \text{ in } D \times \mathbb{Z}_+ \\ u_0(x,i)\big|_{\partial D} = \phi(x,i) \text{ on } \partial D \times \mathbb{Z}_+. \end{cases} \tag{3.67}$$

Let $Y^{x,i}(t)$ be the solution to

$$dY(t) = b(Y(t),i)dt + \sigma(Y(t),i)dW(t),\ t \geq 0 \tag{3.68}$$

with initial condition $x$ and $\tau_D^{x,i} = \inf\{t \geq 0 : Y^{x,i}(t) \notin \mathcal{D}\}$. In view of the Feyman-Kac formula for diffusion processes

$$\begin{aligned} u_0(x,i) =& \mathbb{E}_{x,i}\left[\phi(Y(\tau_D)), i) \exp\left(-\int_0^{\tau_D} q_i(Y(s))ds\right)\right] \\ &- \mathbb{E}_{x,i}\int_0^{\tau_D} \exp\left(-\int_0^t q_i(Y(s))ds\right) f(Y(t),i)dt. \end{aligned} \tag{3.69}$$

Note that in (3.69) and what follows, we drop the superscripts $x$ and $i$ in $Y^{x,i}$ and $\tau_D^{x,i}$ whenever the expectation $\mathbb{E}_{x,i}$ or probability $\mathbb{P}_{x,i}$ is used. By part (2) of Assumption 3.1, $\sup_{x\in\mathcal{D}} \mathbb{E}_{x,i}\tau_D < \infty$ for any $i \in \mathbb{Z}_+$. In view of (3.69), we have

$$|u_0(x,i)| \leq \sup_{x\in\partial\mathcal{D}}\{|\phi(x,i)|\} + \sup_{x\in\mathcal{D}}\{|f(x,i)|\}\sup_{x\in\mathcal{D}}\{\mathbb{E}_{x,i}\tau_D\}. \tag{3.70}$$

Let $n_0$ and $\varepsilon_0$ satisfy (3.62). In particular, for $i > n_0$, $q_i(x) \geq \varepsilon_0 > 0$ in $D$, we can have the following estimate from (3.69):

$$\begin{aligned} |u_0(x,i)| \leq& \mathbb{E}_{x,i}|\phi_i(Y(\tau_D))| + \mathbb{E}_{x,i}\int_0^{\tau_D} \exp\left(-\int_0^t q_i(Y(s))ds\right)|f(Y(t),i)|dt \\ &\leq \sup_{x\in\partial\mathcal{D}}\{|\phi(x,i)|\} + \sup_{x\in\mathcal{D}}\{|f(x,i)|\}\mathbb{E}_{x,i}\int_0^{\tau_D} \exp(-\varepsilon_0 t)dt \\ &\leq \sup_{x\in\partial\mathcal{D}}\{|\phi(x,i)|\} + \sup_{x\in\mathcal{D}}\{|f(x,i)|\}\mathbb{E}_{x,i}\int_0^{\tau_D} \exp(-\varepsilon_0 t)dt \\ &\leq \sup_{x\in\partial\mathcal{D}}\{|\phi(x,i)|\} + \varepsilon_0^{-1}\sup_{x\in\mathcal{D}}\{|f(x,i)|\}. \end{aligned} \tag{3.71}$$

As a result of (3.66), (3.70), and (3.71),

$$\begin{aligned} M_0 := \sup_{(x,i)\in D\times\mathbb{Z}_+} |u_0(x,i)| \leq& \sup_{x\in\partial\mathcal{D},i\in\mathbb{Z}_+}\{|\phi(x,i)|\} + \varepsilon_0^{-1}\sup_{x\in\mathcal{D},i>n_0}\{|f(x,i)|\} \\ &+ \sup_{x\in\mathcal{D},i\leq n_0}\{|f(x,i)|\}\sup_{x\in\mathcal{D},i\leq n_0}\{\mathbb{E}_{x,i}\tau_D\} < \infty. \end{aligned} \tag{3.72}$$

Since $u_0(x,i)$ is continuous in $\overline{\mathcal{D}} \times \mathbb{Z}_+$ and $q_i(x) = \sum_{j\in\mathbb{Z}_+} q_{ij}(x)$ is continuous and bounded in $\overline{\mathcal{D}} \times \mathbb{Z}_+$, it is easy to show that $\sum_{j\in\mathbb{Z}_+} q_{ij} u_0(x,j)$ is continuous in $\overline{\mathcal{D}} \times \mathbb{Z}_+$ and

$$\sup_{(x,i)\in D\times\mathbb{Z}_+} \left| \sum_{j\in\mathbb{Z}_+} q_{ij} u_0(x,j) \right| := \widehat{M}_0 < \infty. \tag{3.73}$$

Thus, for each $i \in \mathbb{Z}_+$, there exists a strong solution $u_1(x,i) \in W^{2,p}_{loc}(D) \cap C(\overline{D})$ to

$$\begin{cases} \mathcal{L}_i u_1(x,i) - q_i(x) u_1(x,i) = -\sum_{j\in\mathbb{Z}_+} q_{ij}(x) u_0(x,j) \text{ in } D \times \mathbb{Z}_+ \\ u_1(x,i)\big|_{\partial D} = \phi(x,i) \text{ on } \partial D \times \mathbb{Z}_+. \end{cases} \tag{3.74}$$

owing to [54, Theorem 9.1.5]. Similar to (3.72), we can use (3.73) to obtain that

$$\sup_{(x,i)\in D\times\mathbb{Z}_+} |u_1(x,i)| := M_1 < \infty. \tag{3.75}$$

Continuing this way, we can define recursively $\{u_{m+1}(x,i)\} \in \mathbb{H}^{2,p}(D)$, the strong solution to

$$\begin{cases} \mathcal{L}_i u_{m+1}(x,i) - q_i(x) u_{m+1}(x,i) = -\sum_{j\in\mathbb{Z}_+} q_{ij}(x) u_m(x,j) \text{ in } D \times \mathbb{Z}_+ \\ u_{m+1}(x,i)\big|_{\partial D} = \phi(x,i) \text{ on } \partial D \times \mathbb{Z}_+. \end{cases} \tag{3.76}$$

By the Feyman-Kac formula,

$$\begin{aligned} u_{m+1}(x,i) =& \mathbb{E}_{x,i}\left[\phi_i(Y(\tau_D))) \exp\left(-\int_0^{\tau_D} q_i(Y(s))ds\right)\right] \\ &+ \mathbb{E}_{x,i} \int_0^{\tau_D} \exp\left(-\int_0^t q_i(Y(s))ds\right) \sum_{j\in\mathbb{Z}_+} q_{ij}(Y(t)) u_m(Y(t),j)dt \\ &- \mathbb{E}_{x,i} \int_0^{\tau_D} \exp\left(-\int_0^t q_i(Y(s))ds\right) f(Y(t),i)dt. \end{aligned} \tag{3.77}$$

Let $\Delta_m(x,i) = u_{m+1}(x,i) - u_m(x,i)$ and

$$\Delta_m^i = \sup\{|\Delta_{m+1}(x,i)| : x \in \mathcal{D}\}$$

It follows from (3.77) that

$$\begin{aligned}
|\Delta_{m+1}(x,i)| =& \mathbb{E}_{x,i} \int_0^{\tau_D} \exp\left(-\int_0^t q_i(Y(s))ds\right) \sum_{j\in\mathbb{Z}_+} q_{ij}(Y(t))|\Delta_m(Y(t),j)|dt \\
&\le \sup_{\{i\in\mathbb{Z}_+\}} \{\Delta_m^i\} \mathbb{E}_{x,i} \int_0^{\tau_D} \exp\left(-\int_0^t q_i(Y(s))ds\right) q_i(Y(t))dt \qquad (3.78)\\
&= \sup_{\{i\in\mathbb{Z}_+\}} \{\Delta_m^i\} \mathbb{E}_{x,i} \left[1 - \exp\left(-\int_0^{\tau_D} q_i(Y(s))ds\right)\right].
\end{aligned}$$

Let

$$p := \max_{\{i\le n_0\}} \mathbb{E}_{x,i} \left[1 - \exp\left(-\int_0^{\tau_D} q_i(Y(s))ds\right)\right] < 1.$$

We have from (3.78) that

$$\max_{\{i\le n_0\}} \{\Delta_{m+1}^i\} \le p \sup_{\{i\in\mathbb{Z}_+\}} \{\Delta_m^i\}. \qquad (3.79)$$

It also follows from (3.78) that

$$\sup_{\{i\in\mathbb{Z}_+\}} \{\Delta_{m+1}^i\} \le \sup_{\{i\in\mathbb{Z}_+\}} \{\Delta_m^i\}. \qquad (3.80)$$

For $i > n_0$, using (3.78) again and then using (3.79) and (3.80), we have

$$\begin{aligned}|\Delta_{m+2}(x,i)| \leq &\max_{\{i\leq n_0\}}\{\Delta^i_{m+1}\}\mathbb{E}_{x,i}\int_0^{\tau_D}\exp\left(-\int_0^t q_i(Y(s))ds\right)\sum_{j\leq n_0} q_{ij}(Y(t))dt \\ &+ \sup_{\{i>n_0\}}\{\Delta^i_{m+1}\}\mathbb{E}_{x,i}\int_0^{\tau_D}\exp\left(-\int_0^t q_i(Y(s))ds\right)\sum_{j>n_0} q_{ij}(Y(t))dt \\ \leq& p\sup_{\{i\in\mathbb{Z}_+\}}\{\Delta^i_{m+1}\}\mathbb{E}_{x,i}\int_0^{\tau_D}\exp\left(-\int_0^t q_i(Y(s))ds\right)\sum_{j\leq n_0} q_{ij}(Y(t))dt \\ &+ \sup_{\{i\in\mathbb{Z}_+\}}\{\Delta^i_{m+1}\}\mathbb{E}_{x,i}\int_0^{\tau_D}\exp\left(-\int_0^t q_i(Y(s))ds\right)\sum_{j>n_0} q_{ij}(Y(t))dt \\ \leq& p\sup_{\{i\in\mathbb{Z}_+\}}\{\Delta^i_{m+1}\}\mathbb{E}_{x,i}\int_0^{\tau_D}\exp\left(-\int_0^t q_i(Y(s))ds\right)\sum_{j\in\mathbb{Z}_+} q_{ij}(Y(t))dt \\ &+ (1-p)\sup_{\{i\in\mathbb{Z}_+\}}\{\Delta^i_{m+1}\}\mathbb{E}_{x,i}\int_0^{\tau_D}\exp\left(-\int_0^t q_i(Y(s))ds\right)\sum_{j>n_0} q_{ij}(Y(t))dt.\end{aligned} \tag{3.81}$$

Let

$$M_D = \sup_{(x,i)\in\mathcal{D}\times N} q_i(x). \tag{3.82}$$

Note that

$$\frac{\sum_{j>n_0} q_{ij}(x)}{q_i(x)} = 1 - \frac{\sum_{j\leq n_0} q_{ij}(x)}{q_i(x)} \leq 1 - \frac{\varepsilon_0}{M_D} := \varepsilon_1 \quad \text{for} \quad i > n_0,$$

which implies that

$$\begin{aligned}&\mathbb{E}_{x,i}\int_0^{\tau_D}\exp\left(-\int_0^t q_i(Y(s))ds\right)\sum_{j>n_0} q_{ij}(Y(t))dt \\ &\leq \varepsilon_1\mathbb{E}_{x,i}\int_0^{\tau_D}\exp\left(-\int_0^t q_i(Y(s))ds\right)q_i(Y(t))dt \\ &\leq \varepsilon_1 \qquad \text{for} \quad i > n_0.\end{aligned} \tag{3.83}$$

In view of (3.81) and (3.83), we have

$$\sup_{\{i>n_0\}} \Delta^i_{m+2} \leq [p+(1-p)\varepsilon_1] \sup_{\{i\in\mathbb{Z}_+\}} \{\Delta^i_{m+1}\}. \tag{3.84}$$

By (3.79) and (3.80),

$$\sup_{\{i\leq n_0\}} \{\Delta^i_{m+2}\} \leq p \max_{\{i\in\mathbb{Z}_+\}} \{\Delta^i_{m+1}\} \leq \max_{\{i\in\mathbb{Z}_+\}} \{\Delta^i_m\}. \tag{3.85}$$

By (3.85) and (3.84),

$$\sup_{\{i\in\mathbb{Z}_+\}} \{\Delta^i_{m+2}\} \leq [p+(1-p)\varepsilon_1] \max_{\{i\in\mathbb{Z}_+\}} \{\Delta^i_m\}. \tag{3.86}$$

In view of (3.72) and (3.73), $\sup_{i\in\mathbb{Z}_+}\{\Delta^i_1\} \leq M_0+M_1 < \infty$. Since $p+(1-p)\varepsilon_1 < 1$, it follows from (3.80) and (3.86) that the series $\sum_{m=1}^{\infty} \sup_{\{i\in\mathbb{Z}_+\}}\{\Delta^i_{m+2}\}$ is convergent. Thus $u_m(x,i)$ converges uniformly in $(x,i)$ to a function $u(x,i)$. For each $i \in \mathbb{Z}_+$, since $q_i(x) = \sum_j q_{ij}(x)$ is continuous, the convergence $\lim_{k\to\infty} \sum_{j<k} q_{ij}(x) = q_i(x)$ is uniform. Thus, it is easy to show that as $m \to \infty$, $\sum_j q_{ij}(x)u_m(x,j)$ converges uniformly to $\sum_j q_{ij}(x)u(x,j)$, (which is also continuous in $x$). Using this uniform convergence, passing the limit in (3.77) we have

$$\begin{aligned} u(x,i) =& \mathbb{E}_{x,i}\left[\phi_i(Y(\tau_D))) \exp\left(-\int_0^{\tau_D} q_i(Y(s))ds\right)\right] \\ &+ \mathbb{E}_{x,i}\int_0^{\tau_D} \exp\left(-\int_0^t q_i(Y(s))ds\right) \sum_{j\in\mathbb{Z}_+} q_{ij}u(Y(t),j)dt \\ &- \mathbb{E}_{x,i}\int_0^{\tau_D} \exp\left(-\int_0^t q_i(Y(s))ds\right) f(Y(t),i)dt. \end{aligned} \tag{3.87}$$

Since $\sum_j q_{ij}(x)u(x,j)$ is continuous in $x$ for each $i \in \mathbb{Z}_+$, the representation (3.87) shows that $u(x,i)$ satisfies

$$\mathcal{L}_i u(x,i) - q_i(x)u(x,i) = f(x,i) - \sum_{j\in\mathbb{Z}_+} q_{ij}(x)u(x,j) \text{ in } D\times\mathbb{Z}_+$$

Since $u_m(x,i) = \phi(x,i)$ on $\partial D \times \mathbb{Z}_+$ for all $m \in \mathbb{Z}_+$, we have $u(x,i) = \phi(x,i)$ on $\partial D \times \mathbb{Z}_+$. The existence of solutions is therefore proved. To prove the uniqueness, it suffices to consider the uniqueness in $\mathbb{H}^p(D)$ of the system

$$\begin{cases} \mathcal{L}_i v(x,i) - q_i(x)v(x,i) + \sum_{j=1}^{\infty} q_{ij}(x)v(x,j) = 0 \text{ in } D \times \mathbb{Z}_+ \\ v(x,i)\big|_{\partial D} = 0 \text{ on } \partial D \times \mathbb{Z}_+. \end{cases} \tag{3.88}$$

Let $\{v(x,i)\} \in \mathbb{H}^p(D)$ be a solution of (3.88). Then we have

$$v(x,i) = -\mathbb{E}_{x,i} \int_0^{\tau_D} \exp\left( - \int_0^t q_i(Y(s))ds \right) \sum_{j \in \mathbb{Z}_+} q_{ij} v(Y(s),j)ds. \tag{3.89}$$

Similar to (3.79), it follows from (3.89) that

$$\sup_{i \le n_0, x \in D} \{|v(x,i)|\} \le p \sup_{i \in \mathbb{Z}_+, x \in D} \{|v(x,i)|\}.$$

Similar to (3.84), the above inequality and (3.89) imply that

$$\sup_{i \in \mathbb{Z}_+, x \in D} \{|v(x,i)|\} \le [p + (1-p)\varepsilon_1] \sup_{i \in \mathbb{Z}_+, x \in D} \{|v(x,i)|\}.$$

Thus $\sup_{i \in \mathbb{Z}_+, x \in D}\{|v(x,i)|\} = 0$, that is, (3.88) has a unique solution. □

**Lemma 3.21.** *Let $D$ be an open and bounded set of $\mathbb{R}^n$. Let $\xi_0 = 0$ and $\xi_k = \inf\{t \ge 0 : \alpha(t) \ne \alpha(\xi_{k-1})\}, k \in \mathbb{Z}_+$. Let $f(x,i)$ and $g(x,i)$ are bounded and measurable functions on $D \times \mathbb{Z}_+$ and $\partial D \times \mathbb{Z}_+$ respectively. Then*

$$\begin{aligned} &\mathbb{E}_{x,i} 1_{\{\xi_1 \le \widetilde{\tau}_D\}} f(X(\xi_1), \alpha(\xi_1)) + \mathbb{E}_x 1_{\{\xi_1 > \widetilde{\tau}_D\}} g(X(\widetilde{\tau}_D), i) \\ =& \mathbb{E}_{x,i} \int_0^{\tau_D} q_{ij}(Y(t)) f(Y(t),j) \exp\left( - \int_0^t q_i(Y(s))ds \right) \\ &\quad + \mathbb{E}_{x,i} g(Y(\tau_D), i) \exp\left( - \int_0^{\tau_D} q_i(Y(t))dt \right). \end{aligned} \tag{3.90}$$

*Proof.* Define

$$\beta^{x,i}(t) = i + \int_0^t \int_{\mathbb{R}} h(Y_t^{x,i}, \beta^{x,i}(t-), z)\mathfrak{p}(dt, dz).$$

Let $\lambda_1^{x,i} = \inf\{t \geq 0 : \beta^{x,i}(t) \neq i\}$. We have that

$$(X^{x,i}(t), \alpha^{x,i}(t)) = (Y^{x,i}(t), \beta^{x,i}(t)) \text{ up to } \lambda_1^{x,i} = \xi_1^{x,i}, \tag{3.91}$$

where $(X^{x,i}(t), \alpha^{x,i}(t))$ is the solution to (1.7) with initial value $(x, i)$ and $\xi_1^{x,i}$ is the first moment of jump for $\alpha^{x,i}(t)$. Thus,

$$\mathbb{P}_{x,i}\{\xi_1 \wedge \widetilde{\tau}_D < \infty\} = \mathbb{P}_{x,i}\{\lambda_1 \leq \tau_D\} \geq \mathbb{P}_{x,i}\{\tau_D < \infty\} = 1. \tag{3.92}$$

In view of Lemma 2.9,

$$\begin{aligned}
&\mathbb{E}_{x,i} 1_{\{\lambda_1 \leq \tau_D\}} f(Y(\lambda_1), \beta(\lambda_1)) + \mathbb{E}_{x,i} 1_{\{\lambda_1 > \tau_D\}} g(Y(\tau_D), i) \\
&= \mathbb{E}_{x,i} \int_0^{\tau_D} q_{ij}(Y(t)) f(Y(t), j) \exp\left(-\int_0^t q_i(Y(s))ds\right) \\
&\quad + \mathbb{E}_{x,i} g(Y(\tau_D), i) \exp\left(-\int_0^{\tau_D} q_i(Y(t))dt\right).
\end{aligned} \tag{3.93}$$

Combining (3.91) and (3.93), we obtain (3.90). □

**Lemma 3.22.** *Let $D$ be an open bounded set in $\mathbb{R}^n$. For any $\varepsilon > 0$, there is an $n_2 = n_2(\varepsilon) > 0$ such that*

$$\mathbb{P}_{x,i}\{\xi_{n_2} \leq \widetilde{\tau}_D\} < \varepsilon$$

*for any $(x, i) \in \mathcal{B} \times \mathbb{Z}_+$. As a result,*

$$\mathbb{P}_{x,i}\{\widetilde{\tau}_D < \infty\} = 1.$$

*Moreover, for any $k > 0$, there is a $T > 0$ such that*

$$\mathbb{P}_{x,i}\{\xi_k \wedge \widetilde{\tau}_D > T\} < \varepsilon.$$

*Proof.* For each $i \in \mathbb{Z}_+$, we have that

$$p_{i,D} := \sup_{x \in \mathcal{D}} \mathbb{E}_{x,i}\tau_D < \infty.$$

By Lemma 3.21, with $M_D$ defined as in (3.82), we have

$$\begin{aligned}\mathbb{P}_{x,i}\{\xi_1 > \widetilde{\tau}_D\} =& \mathbb{E}_{x,i} \exp\left(-\int_0^{\tau_D} q_i(Y(t))dt\right)\\ \geq& \mathbb{E}_{x,i} \exp\left(-M_D\tau_D\right)\\ \geq& \exp\left(-M_D\mathbb{E}_{x,i}\tau_D\right)\\ \geq& \exp\left(-M_D p_{i,D}\right).\end{aligned} \tag{3.94}$$

Let $\tilde{p} := \min_{\{i \leq n_0\}}\{\exp\left(-M_D p_{i,D}\right)\}$. By (3.62), for $i > n_0$,

$$\frac{\sum_{j \leq n_0} q_{ij}(x)}{q_i(x)} \geq \frac{\varepsilon_0}{M_D} > 0, x \in D.$$

Applying Lemma 3.21 again, we have

$$\begin{aligned}\mathbb{P}_{x,i}\{\xi_1 \leq \widetilde{\tau}_D,\, \alpha(\xi_1) \leq n_0\} =& \mathbb{E}_{x,i} \int_0^{\tau_B} \sum_{j \leq n_0} q_{ij}(Y(t)) \exp\left(-\int_0^t q_i(Y(s))ds\right)\\ \geq& \frac{\varepsilon_0}{M_D}\mathbb{E}_{x,i} \int_0^{\tau_B} q_i(Y(t)) \exp\left(-\int_0^t q_i(Y(s))ds\right)\\ =& \frac{\varepsilon_0}{M_D}\mathbb{P}_{x,i}\{\xi_1 \leq \widetilde{\tau}_D\}.\end{aligned} \tag{3.95}$$

By the strong Markov property, (3.94), and (3.95), we have for $i > n_0$ that

$$\begin{aligned}\mathbb{P}_{x,i}\{\xi_1 \le \widetilde{\tau}_D, \xi_2 > \widetilde{\tau}_D\} \ge & \mathbb{P}_{x,i}\{\xi_2 < \widetilde{\tau}_D,\ \xi_1 \ge \widetilde{\tau}_D,\ \alpha(\xi_1) \le n_0\} \\ \ge & \mathbb{P}_{x,i}\{\xi_1 \le \widetilde{\tau}_D,\ \alpha(\xi_1) \le n_0\} \left[\inf_{\{y\in\mathcal{D}, j\le n_0\}} \mathbb{P}_{y,j}\{\xi_1 < \widetilde{\tau}_D\}\right] \\ \ge & \frac{\tilde{p}\varepsilon_0}{M_D}\mathbb{P}_{x,i}\{\xi_1 \le \widetilde{\tau}_D\}.\end{aligned} \tag{3.96}$$

Since $\tilde{p} < 1$ and $\dfrac{\varepsilon_0}{M_D} \ge 1$, we have from (3.95) that

$$\begin{aligned}\mathbb{P}_{x,i}\{\xi_2 > \widetilde{\tau}_D\} \ge & \mathbb{P}\{\xi_1 > \widetilde{\tau}_D\} + \mathbb{P}_{x,i}\{\xi_1 \le \widetilde{\tau}_D, \xi_2 > \widetilde{\tau}_D\} \\ \ge & \mathbb{P}\{\xi_1 > \widetilde{\tau}_D\} + \frac{\tilde{p}\varepsilon_0}{M_D}\mathbb{P}_{x,i}\{\xi_1 \le \widetilde{\tau}_D\} \\ \ge & \frac{\tilde{p}\varepsilon_0}{M_D} \ \text{ for } \ x \in \mathcal{D}, i > n_0.\end{aligned} \tag{3.97}$$

In light of (3.94),

$$\mathbb{P}_{x,i}\{\xi_2 > \widetilde{\tau}_D\} \ge \mathbb{P}\{\xi_1 > \widetilde{\tau}_D\} \ge \tilde{p} \ \text{ for } \ x \in D, i \le n_0. \tag{3.98}$$

Thus, for any $x \in D$ and $i \in \mathbb{Z}_+$, we have

$$\mathbb{P}_{x,i}\{\xi_2 > \widetilde{\tau}_D\} \ge \frac{\tilde{p}\varepsilon_0}{M_D}. \tag{3.99}$$

Using the strong Markov property, we have from (3.99) that

$$\mathbb{P}_{x,i}\{\xi_{2k} \le \widetilde{\tau}_D\} \le \left(1 - \frac{\tilde{p}\varepsilon_0}{M_D}\right)^k. \tag{3.100}$$

By letting $n_2 = 2k_2+1$ with $k_2$ being sufficiently large so that $\left(1 - \dfrac{\tilde{p}\varepsilon_0}{M_D}\right)^{k_2} < \varepsilon$, we complete the proof for the first part of this lemma.

To prove the second part, note that $\mathbb{E}_{x,i}\tau_D \le p_{i,D} < \infty$, thus for any $\varepsilon' > 0$, there is

$T_1 > 0$ such that

$$\begin{aligned}\mathbb{P}_{x,i}\{\xi_1 \wedge \widetilde{\tau}_D \leq T_1\} =& \mathbb{P}_{x,i}\{\lambda_1 \wedge \tau_D \leq T_1\} \\ \geq& \mathbb{P}_{x,i}\{\tau_D \leq T_1\} > 1-\varepsilon' \text{ for all } x \in \mathcal{D}, i \leq n_0.\end{aligned} \tag{3.101}$$

For $i > n_0$, we have

$$\begin{aligned}\mathbb{P}_{x,i}\{\xi_1 \wedge \widetilde{\tau}_D > T\} =& \mathbb{P}_{x,i}\{\tau_D > T, \lambda_1 > T\} \\ \geq& \mathbb{E}_{x,i}\left[\mathbf{1}_{\{\tau_D > T\}} \int_0^T q_i(Y(t)) \exp\left(-\int_0^t q_i(Y(s))ds\right) dt\right] \\ =& \mathbb{E}_{x,i}\left[\mathbf{1}_{\{\tau_D > T\}} \exp\left(-\int_0^T q_i(Y(s))ds\right)\right] \\ \leq& \mathbb{E}_{x,i}\left[\mathbf{1}_{\{\tau_D > T\}} \exp\left(-T\varepsilon_0\right)\right] \;\; (\text{since } \; q_i(x) > \varepsilon \text{ if } x \in \mathcal{D}, i > n_0).\end{aligned} \tag{3.102}$$

Let $T_2 > T_1$ such that $\exp(-T_2\varepsilon_0) < \varepsilon'$. We have from (3.102) that

$$\mathbb{P}_{x,i}\{\xi_1 \wedge \widetilde{\tau}_D \leq T_2\} > 1-\varepsilon' \;\text{ for }\; x \in \mathcal{D}, i > n_0. \tag{3.103}$$

Using (3.101) and (3.103),

$$\mathbb{P}_{x,i}\{\xi_1 \wedge \widetilde{\tau}_D \leq T_2\} > 1-\varepsilon' \;\text{ for }\; x \in \mathcal{D}, i \in \mathbb{Z}_+. \tag{3.104}$$

Using the strong Markov property, it is easy to show that

$$\mathbb{P}_{x,i}\{\xi_k \wedge \widetilde{\tau}_D \leq kT_2\} > (1-\varepsilon')^k \;\text{ for }\; x \in \mathcal{D}, i \in \mathbb{Z}_+. \tag{3.105}$$

By choosing $\varepsilon'$ such that $(1-\varepsilon')^k > 1-\varepsilon$, we obtain the second part of this lemma. □

**Lemma 3.23.** *Let $D \in \mathbb{R}^n$ be a bounded set. For $i_0 \in \mathbb{Z}_+, T > 0, \varepsilon > 0$, there is a $k_0 = k_0(i_0, T, \varepsilon) > 0$ such that*

$$\mathbb{P}_{x,i_0}\{\zeta_{k_0} > T\} < \varepsilon,\, x \in D,$$

*where* $\zeta_k = \inf\{t > 0 : \alpha(t) \geq k\}$.

*Proof.* This lemma is a direct consequence of Lemma 2.12 and the Heine-Borel covering theorem. □

To proceed, we need the following lemma, which is a weak form of Harnack's principle.

**Lemma 3.24.** *Let $D$ be an open bounded set in $\mathbb{R}^n$ with $\partial D \in C^2$ and fix $(x_0, i_0) \in \mathcal{D} \times \mathbb{Z}_+$. Let $B \subset \overline{B} \subset D$ be a ball centered at $x_0$. Then for any $\varepsilon > 0$, there is a $c_0 = c_0(B, i_0, \varepsilon) > 0$ satisfying*

$$u(x, i_0) \leq c_0 u(x_0, i_0) + \varepsilon \sup_{\{(y,i) \in \partial D \times \mathbb{Z}_+\}} \{u(y, i)\}, x \in \overline{B},$$

*where $\{u(x, i)\} \in \mathbb{H}^p(D)$ satisfies*

$$\mathcal{L}_i u(x, i) - q_i(x) u(x, i) + \sum_{j \in \mathbb{Z}_+} q_{ij}(x) u(x, j) = 0 \text{ in } D \times \mathbb{Z}_+.$$

*Proof.* Let $\phi(x, i) = u(x, i)|_{\partial D}$ and

$$\zeta_k = \inf\{t > 0 : \alpha(t) \geq k\}.$$

Let

$$u_k(x, i) = \begin{cases} u(x, i) & \text{if } i < k \\ 0 & \text{if } i \geq k. \end{cases}$$

By Itô's formula,

$$\begin{aligned} &\mathbb{E}_{x,i} u_k(X(\widetilde{\tau}_D \wedge \zeta_k \wedge t), \alpha(\widetilde{\tau}_D \wedge \zeta_k \wedge t)) \\ =& u_k(x, i) + \mathbb{E}_{x,i} \int_0^{\widetilde{\tau}_D \wedge \zeta_k \wedge t} \mathcal{L} u_k(X(s), \alpha(s)) ds \\ =& u_k(x, i) - \mathbb{E}_{x,i} \int_0^{\widetilde{\tau}_D \wedge \zeta_k \wedge t} \sum_{j \geq k} q_{\alpha(s),j}(X(s)) u_k(X(s), j) ds. \end{aligned} \tag{3.106}$$

Letting $k \to \infty$ and then $t \to \infty$, we obtain from the dominated convergence theorem that

$$u(x,i) = \mathbb{E}_{x,i}\phi(X(\widetilde{\tau}_D), \alpha(\widetilde{\tau}_D)). \tag{3.107}$$

As a result of Lemmas 3.22 and 3.23, there is a $k_1 = k_1(i_0, \varepsilon) \in \mathbb{Z}_+$ such that

$$\mathbb{P}_{x,i}\{\widetilde{\tau}_D > \xi_{k_1}\} < \varepsilon. \tag{3.108}$$

In view of (3.107) and (3.108),

$$\begin{aligned} u(x,i_0) =& \mathbb{E}_{x,i_0}\mathbf{1}_{\{\widetilde{\tau}_D < \xi_{k_1}\}}\phi(X(\widetilde{\tau}_D), \alpha(\widetilde{\tau}_D)) + \mathbb{E}_{x,i_0}\mathbf{1}_{\{\widetilde{\tau}_D > \xi_{k_1}\}}\phi(X(\widetilde{\tau}_D), \alpha(\widetilde{\tau}_D)) \\ \leq& \mathbb{E}_{x,i_0}\mathbf{1}_{\{\widetilde{\tau}_D < \xi_{k_1}\}}\phi(X(\widetilde{\tau}_D), \alpha(\widetilde{\tau}_D)) + \varepsilon \sup_{(y,j)\in\partial D\times\mathbb{Z}_+}\{\phi(y,j)\}. \end{aligned} \tag{3.109}$$

Let

$$\tilde{u}(x,i) = \mathbb{E}_{x,i}\mathbf{1}_{\{\widetilde{\tau}_D < \xi_{k_1}\}}\phi(X(\widetilde{\tau}_D), \alpha(\widetilde{\tau}_D))$$

for $i < k$. The process $\{(X(t), \alpha(t)), 0 \leq t < \xi_{k_1}\}$ can be considered as a switching diffusion process on $\mathbb{R}^n \times \{1, \ldots, k_1 - 1\}$ with lifetime $\xi_{k_1}$. Its generator is

$$\tilde{\mathcal{L}}_i f(x,i) = \mathcal{L}_i f(x,i) - q_i(x)u(x,i) + \sum_{j<k_1} q_{ij}(x) f(x,j),$$

for $i = 1, \ldots, k-1$. Then [8, Theorem 3.6] reveals that $\tilde{u}(x,i)$ satisfying

$$\begin{cases} \tilde{\mathcal{L}}_i\tilde{u}(x,i) = 0 \text{ in } D \times \{1, \ldots, k_1 - 1\} \\ \tilde{u}(x,i)\big|_{\partial D} = \phi(x,i) \text{ on } \partial D \times \{1, \ldots, k_1 - 1\}. \end{cases} \tag{3.110}$$

By the Harnack principle for weakly coupled elliptic systems (see e.g., [7]), there is a $c_0 = c_0(k_1)$ such that

$$\tilde{u}(x,i_0) \leq c_0\tilde{u}(x_0,i_0) \leq c_0 u(x_0,i_0) \tag{3.111}$$

The desired result follows from (3.109) and (3.111). □

**Remark 3.25.** In (3.69) and (3.77), we apply the Feyman-Kac formula for functions in the class $W^{2,p}_{loc}(D)\cap C(\overline{D})$ rather than $C^2(D)$. Feyman-Kac formula is proved using Itô's formula, which is usually stated for $C^2$-functions. However, Itô's formula also holds for diffusion processes with functions in $W^{2,p}_{loc}(D)\cap C(\overline{D})$ when $p>n$. The proof for this claim can be found in [27, Theorem 2.10.2]. With a careful consideration, we can generalize the result for diffusion processes to switching diffusion processes in which the switching has a finite state space. Thus, (3.106) holds as long as $u(\cdot,i)\in W^{2,p}_{loc}(D)\cap C(\overline{D})$.

*Proof of Theorem* 3.19. Let $k_0\in\mathbb{Z}_+$ sufficiently large such that $D_1\subset\{x\in\mathbb{R}^n:|x|<k_0\}$. For $k>k_0$, define $D_k=D\cap\{x\in\mathbb{R}^n:|x|<k\}$. By (3.107), $u_k(x,i):=\mathbb{E}_{x,i}\widetilde{\tau}_{D_k}$ satisfies the equation

$$\begin{cases} \mathcal{L}_i u_k(x,i)-q_i(x)u_k(x,i)+\sum_{j\in\mathbb{Z}_+}q_{ij}(x)u_k(x,j)=-1 \text{ in } D_k\times\mathbb{Z}_+ \\ u_k(x,i)\big|_{\partial D}=1 \text{ on } \partial D_k\times\mathbb{Z}_+. \end{cases} \tag{3.112}$$

Let $B_1\subset B_2$ be two balls in $D$ and fix $(x_0,i_0)\in B_1\times\mathbb{Z}_+$ and let $k_1>k_0$ be such that $B_2\subset D_{k_1}$. Suppose that $\mathbb{E}_{x,i}\widetilde{\tau}_D<M$ for any $(x,i)\in B_2\times\mathbb{Z}_+$. Then $u_k(x,i)<M$ for $k>k_0$ and $(x,i)\in B_2\times\mathbb{Z}_+$. Let $v_{k,m}=u_k(x,i)-u_m(x,i)$ for $k>m>k_1$, we have

$$\mathcal{L}_i v_{k,m}(x,i)-q_i(x)v_{k,m}(x,i)+\sum_{j\in\mathbb{Z}_+}q_{ij}(x)v_{k,m}(x,j)=0 \text{ in } B_2\times\mathbb{Z}_+. \tag{3.113}$$

By Lemma 3.24, for any $\varepsilon>0$, there is a $c_0>0$ such that

$$\begin{aligned} v_{k,m}(x,i_0)\leq& c_0v_{k,m}(x_0,i_0)+\varepsilon\sup\{v_{k,m}(y,j):(y,j)\in B_2\times\mathbb{Z}_+\} \\ \leq& c_0v_{k,m}(x_0,i_0)+M\varepsilon \text{ for any } x\in B_1. \end{aligned} \tag{3.114}$$

For any $\varepsilon > 0$, since $u_k(x_0, i_0) = \mathbb{E}_{x_0,i_0}\widetilde{\tau}_{D_k} \to \mathbb{E}_{x_0,i_0}\widetilde{\tau}_D$ as $k \to \infty$, there exists $k_2 = k_2(\varepsilon)$ such that $c_0 v_{k,m}(x_0, i_0) = c_0[u_k(x_0, i_0) - u_m(x_0, i_0)] < \varepsilon$ for any $k > m > k_2$. In view of (3.114),

$$v_{k,m}(x, i_0) \leq (M + 1)\varepsilon \ \text{ for any } (x, i_0) \in B_1 \times \mathbb{Z}_+, k > m > k_2 \tag{3.115}$$

Thus, $u_k(x, i_0)$ converges uniformly in each compact subset of $D$. The limit $u(x, i_0)$ is therefore continuous for any $i_0$. Now, let $\phi(x, i) = u(x, i)|_{\partial B_2}$. Since $\phi(x, i)$ is continuous and uniformly bounded, by Lemma 3.20, for each $i \in \mathbb{Z}_+$, there exists $\{\tilde{u}(x, i)\} \in \mathbb{H}^p(B_2)$ satisfying

$$\begin{cases} \mathcal{L}_i\tilde{u}(x, i) - q_i(x)\tilde{u}_k(x, i) + \sum_{j\in\mathbb{Z}_+} q_{ij}(x)\tilde{u}(x, j) = -1 \text{ in } B_2 \times \mathbb{Z}_+ \\ \tilde{u}(x, i)\big|_{\partial D} = \phi(x, i) \text{ on } \partial B_2 \times \mathbb{Z}_+. \end{cases} \tag{3.116}$$

Similar to (3.107), by applying Itô's formula we have that

$$\begin{aligned} \tilde{u}(x, i) =& \mathbb{E}_{x,i}\widetilde{\tau}_{B_2} + \mathbb{E}_{x,i}\phi(X(\widetilde{\tau}_{B_2}), \alpha(\widetilde{\tau}_{B_2})) \\ =& \mathbb{E}_{x,i}\widetilde{\tau}_{B_2} + \mathbb{E}_{x,i}\mathbb{E}_{X(\widetilde{\tau}_{B_2}),\alpha(\widetilde{\tau}_{B_2})}\widetilde{\tau}_D \\ =& \mathbb{E}_{x,i}\widetilde{\tau}_D \ \text{ (due to the strong Markov property)} \\ =& u(x, i). \end{aligned}$$

The proof is concluded. □

*Proof of Theorem* 3.18. After having Lemma 3.24, we adapt the proof of [24, Theorem 3.10] to obtain the desired result. First, suppose that (3.63) has a unique solution in $\mathbb{H}^p(D)$ for some $p > 0$ given that $f(x, i)$ is continuous and bounded on $D \times \mathbb{Z}_+$. We define $D_k$ as in the proof of Theorem 3.19 and $v_k(x, i) := \mathbb{P}_{x,i}\{X(\widetilde{\tau}_{D_k}) \in \partial D\}$. By (3.107), $v_k(x, i)$ is the strong solution to

$$\begin{cases} \mathcal{L}_i v_k(x,i) - q_i(x) v_k(x,i) + \sum_{j\in\mathbb{Z}_+} q_{ij}(x) v_k(x,j) = 0 \text{ in } D_k \times \mathbb{Z}_+ \\ v_k(x,i)\big|_{\partial D} = 1 \text{ on } \partial D \times \mathbb{Z}_+. \\ v_k(x,i)\big|_{\partial D} = 0 \text{ on } \{y \in \mathbb{R}^n : |y| = k\} \times \mathbb{Z}_+. \end{cases} \tag{3.117}$$

By the definition of $v_k(x,i)$, we have that

$$\lim_{k\to\infty} v_k(x,i) = v(x,i) := \mathbb{P}_{x,i}\{\widetilde{\tau}_D < \infty\} \tag{3.118}$$

On the other hand, owing to Lemma 3.24, we can use arguments in the proof of Theorem 3.19 to show that $\{v(x,i)\} \in \mathbb{H}^p(D)$ is the solution to

$$\begin{cases} \mathcal{L}_i v(x,i) - q_i(x) v(x,i) + \sum_{j\in\mathbb{Z}_+} q_{ij}(x) v(x,j) = 0 \text{ in } D \times \mathbb{Z}_+ \\ v(x,i)\big|_{\partial D} = 1 \text{ on } \partial D \times \mathbb{Z}_+. \end{cases} \tag{3.119}$$

Clearly, $v(x,i) \equiv 1$ is the solution to (3.120). By the uniqueness of solutions among the class $\mathbb{H}^p(D)$, we have

$$\mathbb{P}_{x,i}\{\widetilde{\tau}_D < \infty\} = v(x,i) \equiv 1.$$

Now, suppose that $\mathbb{P}_{x,i}\{\widetilde{\tau}_D < \infty\} \equiv 1$ and (3.63) has two solutions $\{v^{(1)}(x,i)\}$ and $\{v^{(2)}(x,i)\}$ for the same $f(x,i)$ being continuous and bounded in $\partial D \times \mathbb{Z}_+$. Let $v^{(3)}(x,i) = v^{(1)}(x,i) - v^{(2)}(x,i)$. Then $\{v^{(3)}(x,i)\} \in \mathbb{H}^p(D)$ and satisfies

$$\begin{cases} \mathcal{L}_i v^{(3)}(x,i) - q_i(x) v^{(3)}(x,i) + \sum_{j\in\mathbb{Z}_+} q_{ij}(x) v^{(3)}(x,j) = 0 \text{ in } D \times \mathbb{Z}_+ \\ v^{(3)}(x,i)\big|_{\partial D} = 0 \text{ on } \partial D \times \mathbb{Z}_+. \end{cases} \tag{3.120}$$

Let $M^{(3)} = \sup_{(x,i)\in D\times\mathbb{Z}_+}\{v^{(3)}(x,i)\}$. In view of (3.106), for $k > k_0 \vee |x|$, we have

$$|v^{(3)}(x,i)| = \left|\mathbb{E}_{x,i}\mathbf{1}_{\{|X(\widetilde{\tau}_{D_k})|=k\}}v^{(3)}(X(\widetilde{\tau}_{D_k}),\alpha(\widetilde{\tau}_{D_k}))\right| \le M^{(3)}[1-\mathbb{P}_{x,i}\{X(\widetilde{\tau}_{D_k})\in\partial D\}]$$

Letting $k\to\infty$ and using $\mathbb{P}_{x,i}\{X(\widetilde{\tau}_{D_k})\in\partial D\}\to\mathbb{P}_{x,i}\{\widetilde{\tau}_D<\infty\}=1$ as $k\to\infty$, we obtain $v^{(3)}(x,i)\equiv 0$. □

# CHAPTER 4 STABILITY AND INSTABILITY

In this chapter, we consider stability and instability of switching diffusions. Although our results hold true for past-dependent switching diffusions, we restrict our consideration to the case of past-independent switching diffusions in order to elaborate the main ideas of our results. Comments on stability and instability of past-independent switching diffusions are given in Section 4.3.

## 4.1 Formulation and Auxiliary Results

Consider the two-component process $(X(t), \alpha(t))$, where $\alpha(t)$ is a pure jump process taking value in $\mathbb{Z}_+ = \mathbb{N} \setminus \{0\} = \{1, 2, \dots\}$, the set of positive integers, and $X(t) \in \mathbb{R}^n$ satisfies

$$dX(t) = b(X(t), \alpha(t))dt + \sigma(X(t), \alpha(t))dW(t). \tag{4.1}$$

We assume that the jump intensity of $\alpha(t)$ depends on the current state of $X(t)$, that is, there are functions $q_{ij}(\cdot) : \mathbb{R}^n \to \mathbb{R}$ for $i, j \in \mathbb{Z}_+$ satisfying

$$\begin{aligned}&\mathbb{P}\{\alpha(t+\Delta) = j|\alpha(t) = i, X(s), \alpha(s), s \le t\} = q_{ij}(X(t))\Delta + o(\Delta) \text{ if } i \neq j \quad \text{and}\\ &\mathbb{P}\{\alpha(t+\Delta) = i|\alpha(t) = i, X(s), \alpha(s), s \le t\} = 1 - q_i(X(t))\Delta + o(\Delta).\end{aligned} \tag{4.2}$$

Throughout this paper, $q_{ij}(x) \ge 0$ for each $i \neq j$ and $\sum_{j\in\mathbb{Z}_+} q_{ij}(x) = 0$ for each $i$ and all $x \in \mathbb{R}^n$. Denote $q_i(x) = \sum_{j=1, j\neq i}^{\infty} q_{ij}(x)$ (so $q_{ii}(x) = -q_i(x)$). and $Q(x) = (q_{ij}(x))_{\mathbb{Z}_+\times\mathbb{Z}_+}$. The process $\alpha(t)$ can be defined rigorously as the solution to a stochastic differential equation with respect to a Poisson random measure. For each function $x \in \mathbb{R}^n, i \in \mathbb{Z}_+$, let $\Delta_{ij}(x), j \neq i$ be the consecutive left-closed, right-open intervals of the real line, each having length $q_{ij}(x)$.

That is,

$$\Delta_{i1}(x) = [0, q_{i1}(x)),$$
$$\Delta_{ij}(x) = \Big[ \sum_{k=1,k\neq i}^{j-1} q_{ik}(x), \sum_{k=1,k\neq i}^{j} q_{ik}(x) \Big), j > 1, j \neq i.$$

Define $h : \mathbb{R}^n \times \mathbb{Z}_+ \times \mathbb{R} \mapsto \mathbb{R}$ by $h(x, i, z) = \sum_{j=1,j\neq i}^{\infty} (j - i)\mathbf{1}_{\{z\in\Delta_{ij}(x)\}}$. The process $\alpha(t)$ can be defined as the solution to

$$d\alpha(t) = \int_{\mathbb{R}} h(X_t, \alpha(t-), z)\mathfrak{p}(dt, dz), \tag{4.3}$$

where $a(t-) = \lim_{s\to t^-} \alpha(s)$ and $\mathfrak{p}(dt, dz)$ is a Poisson random measure with intensity $dt \times \mathfrak{m}(dz)$ and $\mathfrak{m}$ is the Lebesgue measure on $\mathbb{R}$ such that $\mathfrak{p}(dt, dz)$ is independent of the Brownian motion $W(\cdot)$. The pair $(X(t), \alpha(t))$ is therefore a solution to

$$\begin{cases} dX(t) = b(X(t), \alpha(t))dt + \sigma(X(t), \alpha(t))dW(t) \\ d\alpha(t) = \displaystyle\int_{\mathbb{R}} h(X(t), \alpha(t-), z)\mathfrak{p}(dt, dz). \end{cases} \tag{4.4}$$

A strong solution to (4.4) on $[0, T]$ with initial data $(x, i) \in \mathbb{R}^n \times \mathbb{Z}_+$ is an $\mathcal{F}_t$-adapted process $(X(t), \alpha(t))$ such that

- $X(t)$ is continuous and $\alpha(t)$ is cadlag (right continuous with left limits) with probability 1 (w.p.1).
- $X(0) = x$ and $\alpha(0) = i_0$
- $(X(t), \alpha(t))$ satisfies (4.4) for all $t \in [0, T]$ w.p.1.

Let $f(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{Z}_+ \mapsto \mathbb{R}$ be twice continuously differentiable in $x$. We define the operator

$\mathcal{L}f(\cdot,\cdot):\mathbb{R}^n\times\mathbb{Z}_+\mapsto\mathbb{R}$ by

$$\begin{aligned}\mathcal{L}f(x,i)=&[\nabla f(x,i)]^\top b(x,i)+\frac{1}{2}\operatorname{tr}\Big(\nabla^2 f(x,i)A(x,i)\Big)+\sum_{j=1,j\neq i}^{\infty}q_{ij}(x)\big[f(x,j)-f(x,i)\big]\\=&\sum_{k=1}^{n}b_k(x,i)f_k(x,i)+\frac{1}{2}\sum_{k,l=1}^{n}a_{kl}(x,i)f_{kl}(x,i)+\sum_{j=1,j\neq i}^{\infty}q_{ij}(x)\big[f(x,j)-f(x,i)\big],\end{aligned}\tag{4.5}$$

where $\nabla f(x,i)=(f_1(x,i),\dots,f_n(x,i))\in\mathbb{R}^{1\times n}$ and $\nabla^2 f(x,i)=(f_{ij}(x,i))_{n\times n}$ are the gradient and Hessian of $f(x,i)$ with respect to $x$, respectively, with

$$\begin{aligned}&f_k(x,i)=(\partial/\partial x_k)f(x,i),\ \ f_{kl}(x,i)=(\partial^2/\partial x_k\partial x_l)f(x,i),\quad\text{and}\\&A(x,i)=(a_{ij}(x,i))_{n\times n}=\sigma(x,i)\sigma^\top(x,i),\end{aligned}$$

where $z^\top$ denotes the transpose of $z$. If $(X(t),\alpha(t))$ satisfies (4.4), then by modifying the proof of [47, Lemma 3, p.104], we have the generalized Itô formula:

$$f(X(t),\alpha(t))-f(X(0),\alpha(0))=\int_0^t\mathcal{L}f(X(s),\alpha(s-))ds+M_1(t)+M_2(t)$$

where $M_1(\cdot)$ and $M_2(\cdot)$ are two local martingales defined by

$$\begin{aligned}M_1(t)=&\int_0^t\nabla f(X(s),\alpha(s-))\sigma(X(s),\alpha(s-))dW(s),\\M_2(t)=&\int_0^t\int_{\mathbb{R}}\big[f\big(X(s),\alpha(s-)+h(X(s),\alpha(s-),z)\big)-f(X(s),\alpha(s-))\big]\mu(ds,dz),\end{aligned}\tag{4.6}$$

and $\mu(ds,dz)$ is the compensated Poisson random measure given by

$$\mu(ds,dz)=\mathfrak{p}(ds,dz)-m(dz)ds.$$

Throughout this chapter, we assume that either one of the following assumptions are satisfied. Under either of them, it is proved in Chapter 2 that (4.4) has a unique solution with given initial data. Moreover, the solution is a Markov-Feller process.

**Assumption 4.1.**

1. For each $i \in \mathbb{Z}_+$, $H > 0$, there is a positive constant $L_{i,H}$ such that

$$|b(x,i) - b(y,i)| + |\sigma(y,i) - \sigma(x,i)| \le L_{i,H}|x - y|$$

   if $x, y \in \mathbb{R}^n$ and $|x|$, $|y| \le H$.

2. For each $i \in \mathbb{Z}_+$, there is a positive constant $\widetilde{L}_i$ such that

$$|b(x,i)| + |\sigma(x,i)| \le \widetilde{L}_i(|x| + 1).$$

3. $q_{ij}(x)$ is continuous in $x \in \mathbb{R}^n$ for each $(i,j) \in \mathbb{Z}_+^2$. Moreover,

$$M := \sup_{x \in \mathbb{R}^n, i \in \mathbb{Z}_+} \{|q_i(x)|\} < \infty.$$

**Assumption 4.2.**

1. For each $i \in \mathbb{Z}_+$, $H > 0$, there is a positive constant $L_{i,H}$ such that

$$|b(x,i) - b(y,i)| + |\sigma(x,i) - \sigma(y,i)| \le L_{i,H}|x - y|$$

   if $x, y \in \mathbb{R}^n$ and $|x|, |y| \le H$.

2. There is a positive constant $\widetilde{L}$ such that

$$|b(x,i)| + |\sigma(x,i)| \le \widetilde{L}(|x| + 1).$$

3. $q_{ij}(x)$ is continuous in $x \in \mathbb{R}^n$ for each $(i,j) \in \mathbb{Z}_+^2$. Moreover, for any $H > 0$,

$$M_H := \sup_{x \in \mathbb{R}^n, |x| \le H, i \in \mathbb{Z}_+} \{|q_i(x)|\} < \infty.$$

We suppose throughout this chapter that $b(0,i) = 0$ and $\sigma(0,i) = 0$ for $i \in \mathbb{Z}_+$ and give the following definitions of stability.

**Definition 4.1.** *The trivial solution $X(t) \equiv 0$ is said to be*

- *stable in probability, if for any $h > 0$,*

$$\lim_{x\to 0} \inf_{i\in\mathbb{Z}_+} \mathbb{P}_{x,i}\big\{X(t) \leq h \,\forall\, t \geq 0\big\} = 1.$$

- *asymptotic stable in probability, if it is stable in probability and*

$$\lim_{x\to 0} \inf_{i\in\mathbb{Z}_+} \mathbb{P}_{x,i}\left\{\lim_{t\to\infty} X(t) = 0\right\} = 1.$$

We state a general result that can be proved by well-known arguments; see [60, Section 7.2].

**Theorem 4.2.** *Let $D$ be a neighborhood of $0 \in \mathbb{R}^n$. Suppose there exist three functions $V(x,i) : D \times \mathbb{Z} \mapsto \mathbb{R}_+$, $\mu_1(x) : D \mapsto \mathbb{R}_+$, $\mu_2(x) : D \mapsto \mathbb{R}_+$ such that*

- *$\mu_1(x), \mu_2(x)$ are continuous on $D$, $\mu_k(x) = 0$ if and only if $x = 0$ for $k = 1, 2$;*
- *$V(x,i)$ is continuous on $D$ and twice continuously differentiable in $\mathcal{D} \setminus \{0\}$ for each $i \in \mathbb{Z}_+$;*
- *$\mu_1(x) \leq V(x,i)$ for any $(x,i) \in D \times \mathbb{Z}_+$.*

*Then the following conclusions hold.*

- *if $\mathcal{L}V(x,i) \leq 0$ for any $(x,i) \in D \times \mathbb{Z}_+$, the trivial solution is stable in probability.*
- *if $\mathcal{L}V(x,i) \leq -\mu_2(x)$ for any $(x,i) \in D \times \mathbb{Z}_+$ the trivial solution is asymptotically stable in probability.*

Let $\widehat{\alpha}(t)$ be the Markov chain with bounded generator $Q(0)$ and transition probability $\widehat{p}_{ij}(t)$

**Definition 4.3.** *The Markov chain $\widehat{\alpha}(t)$ is said to be*

- *ergodic, if it has an invariant probability measure* $\boldsymbol{\nu} = (\nu_1, \nu_2, \dots)$ *and*

$$\lim_{t\to\infty} \widehat{p}_{ij}(t) = \nu_j \text{ for any } i, j \in \mathbb{Z}_+$$

*or equivalently,*

$$\lim_{t\to\infty} \sum_{j\in\mathbb{Z}_+} |\widehat{p}_{ij}(t) - \nu_j| = 0 \text{ for any } i \in \mathbb{Z}_+,$$

- *strongly ergodic, if*

$$\lim_{t\to\infty} \sup_{i\in\mathbb{Z}_+} \left\{ \sum_{j\in\mathbb{Z}_+} |\widehat{p}_{ij}(t) - \nu_j| \right\} = 0.$$

- *strongly exponentially ergodic, if there exist* $C > 0$ *and* $\lambda > 0$ *such that*

$$\sum_{j\in\mathbb{Z}_+} |\widehat{p}_{ij}(t) - \nu_j| \le Ce^{-\lambda t} \text{ for any } i \in \mathbb{Z}_+, t \ge 0. \tag{4.7}$$

We refer to [1] for some properties and sufficient conditions for the aforementioned ergodicity.

## 4.2 Certain Practical Conditions for Stability and Instability

For each $h > 0$, denote by $B_h \subset \mathbb{R}^n$ the open ball centered at 0 with radius $h$. Throughout this section, let $D$ be a neighborhood of 0 satisfying $D \subset B_1$. We also denote by $\widehat{\alpha}(t)$ the continuous-time Markov chain with generator $Q(0)$. Denote by $\mathcal{L}_i$ the generator of the diffusion when the discrete component is in state $i$, that is,

$$\mathcal{L}_i V(x) = \nabla V(x) b(x,i) + \frac{1}{2} \operatorname{tr}\Big(\nabla^2 V(x) A(x,i)\Big).$$

We first state a theorem, which generalizes [26, Theorem 4.3], a result for switching diffusions when the switching takes values in a finite set.

**Theorem 4.4.** *Suppose that the Markov chain $\widehat{\alpha}(t)$ is strongly exponentially ergodic with invariant probability measure $\boldsymbol{\nu} = (\nu_1, \nu_2, \dots)$ and that*

$$\sup_{i\in\mathbb{Z}_+} \sum_{j\neq i} |q_{ij}(x) - q_{ij}(0)| \to 0 \;\; \textit{as} \;\; x \to 0. \tag{4.8}$$

*Let $D$ be a neighborhood of $0$ and $V : D \mapsto \mathbb{R}_+$ satisfying that $V(x) = 0$ if and only if $x = 0$ and that $V(x)$ is continuous on $D$, twice continuously differentiable in $D \setminus \{0\}$. Suppose that there is a bounded sequence of real numbers $\{c_i : i \in \mathbb{Z}_+\}$ such that*

$$\mathcal{L}_i V(x) \leq c_i V(x) \, \forall \, x \in D \setminus \{0\}. \tag{4.9}$$

*Then, if $\sum_{i\in\mathbb{Z}_+} c_i\nu_i < 0$, the trivial solution is asymptotic stable in probability.*

*Proof.* Let $\lambda = -\sum_{i\in\mathbb{Z}_+} c_i\nu_i$. Since $\sum_{i\in\mathbb{Z}_+} \nu_i = 1$, we have $\sum_{i\in\mathbb{Z}_+}(c_i + \lambda)\nu_i = 0$. Since $\widehat{\alpha}(t)$ is strongly exponentially ergodic, it follows from Lemma C.1 that there exists a bounded sequence of real numbers $\{\gamma_i : i \in \mathbb{Z}_+\}$ such that

$$\sum_{j\in\mathbb{Z}_+} q_{ij}(0)\gamma_j = \lambda + c_i \text{ for any } i \in \mathbb{Z}_+ \tag{4.10}$$

Since $\sum_{j\in\mathbb{Z}_+} q_{ij}(0) = 0$ for any $i \in \mathbb{Z}_+$ it follows from (4.10) that

$$\sum_{j\in\mathbb{Z}_+} q_{ij}(0)\gamma_j = \sum_{j\in\mathbb{Z}_+} q_{ij}(0)(1 - p\gamma_j) = -p(\lambda + c_i) \text{ for any } i \in \mathbb{Z}_+ \tag{4.11}$$

Since $\{\gamma_i\}$ is bounded, we can choose $p \in (0, 1)$ such that

$$p|\gamma_i| \leq \min\{0.25\lambda, 0.5\} \tag{4.12}$$

In view of (4.8) and (4.12), there is an $h > 0$ sufficiently small such that

$$\sum_{j\in\mathbb{Z}_+} (1 - p\gamma_j)|q_{ij}(x) - q_{ij}(0)| < \frac{p\lambda}{4} \quad \forall \, x \in B_h. \tag{4.13}$$

Define the function $U(x,i) : B_h \times \mathbb{Z}_+ \mapsto \mathbb{R}_+$ by $U(x,i) = (1-p\gamma_i)V^p(x)$. By Itô's formula, (4.8), (4.11), and (4.13), we have

$$\begin{aligned}\mathcal{L}U(x,i) =& p(1-p\gamma_i)V^{p-1}\mathcal{L}_iV(x) - \frac{p(1-p)}{2}V^{p-2}\left|V_x(x)\sigma(x,i)\right|^2 + V^p(x)\sum_{j\in\mathbb{Z}_+}(1-p\gamma_j)q_{ij}(x)\\ \le& c_ip(1-p\gamma_i)V^{p-1} + V^p(x)\sum_{j\in\mathbb{Z}_+}(1-p\gamma_j)q_{ij}(0) + V^p(x)\sum_{j\in\mathbb{Z}_+}(1-p\gamma_j)|q_{ij}(x)-q_{ij}(0)|\\ \le& c_ip(1-p\gamma_i)V^{p-1} - p(\lambda+c_i)V^p(x) + V^p(x)\sum_{j\in\mathbb{Z}_+}(1-p\gamma_j)|q_{ij}(x)-q_{ij}(0)|\\ \le& p(-\lambda-p\gamma_i)V^p(x) + V^p(x)\sum_{j\in\mathbb{Z}_+}(1-p\gamma_j)|q_{ij}(x)-q_{ij}(0)|\\ \le& -0.75p\lambda V^p(x) + 0.25p\lambda V^p(x) = -0.5p\lambda V^p(x) \text{ for } (x,i)\in B_h\times\mathbb{Z}_+.\end{aligned} \tag{4.14}$$

By Theorem 4.2, it follows from (4.14) that the trivial solution is asymptotically stable.

□

The hypothesis of this theorem seems to be restrictive. It requires the strongly exponential ergodicity of $Q(0)$ and the uniform convergence to 0 of the sum $\sum_{j\neq i}|q_{ij}(x)-q_{ij}(0)|$. To treat cases in which $Q(0)$ is strongly ergodic (not exponentially ergodic) or even only ergodic, as well as to relax the condition (4.8), we need a more complicated method. Our method, which is inspired by the idea in [6], utilizes the ergodicity of $Q(0)$ and the analysis of the Laplace transform. Similar techniques of using the Laplace transform can also be seen in the large deviations theory and related applications [5, 64]. We also take a step further by estimating the pathwise rate of convergence of solutions.

Let $\Gamma$ be a family of increasing and differentiable functions $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $g(y) = 0$ iff $y = 0$ and $\frac{dy}{dy}$ is bounded on $[0,1]$. Since $\frac{dg}{dy}(y)$ is bounded on $[0,1]$ and $g(0) = 0$, it is easy to show that the function

$$G(y) := -\int_y^1 \frac{dz}{g(z)} \quad \text{on } [0,1] \tag{4.15}$$

is non-positive and strictly decreasing and $\lim_{y\to 0} G(y) = -\infty$. Its inverse $G^{-1} : (-\infty, 0] \mapsto (0,1]$ satisfies

$$\lim_{t\to\infty} G^{-1}(-t) = 0.$$

We state some assumptions to be used in what follows; we will also provide some lemmas whose proofs are relegated to the appendix.

**Assumption 4.3.** *There are functions $g \in \Gamma$, $V : D \mapsto \mathbb{R}_+$ such that*

- $V(x) = 0$ *if and only if* $x = 0$
- $V(x)$ *is continuous on* $D$ *and twice continuously differentiable in* $D \setminus \{0\}$.
- *there is a bounded sequence of real numbers* $\{c_i : i \in \mathbb{Z}_+\}$ *such that*

$$\mathcal{L}_i V(x) \le c_i g(V(x)) \, \forall x \in D \setminus \{0\}. \tag{4.16}$$

**Lemma 4.5.** *Under Assumption 4.3, For any $\varepsilon, T, h > 0$, there exists an $\tilde{h} = \tilde{h}(\varepsilon, T, h)$ such that*

$$\mathbb{P}_{x,i}\{\tau_h \le T\} < \varepsilon, \quad \textit{for all } (x,i) \in B_{\tilde{h}} \times \mathbb{Z}_+$$

*where* $\tau_h = \inf\{t \ge 0 : |X(t)| \ge h\}$.

**Lemma 4.6.** *Let $Y$ be a random variable, $\theta_0 > 0$ a constant, and suppose*

$$\mathbb{E}\exp(\theta_0 Y) + \mathbb{E}\exp(-\theta_0 Y) \le K_1.$$

*Then the log-Laplace transform* $\phi(\theta) = \ln \mathbb{E} \exp(\theta Y)$ *is twice differentiable on* $\left[0, \frac{\theta_0}{2}\right)$ *and*

$$\frac{d\phi}{d\theta}(0) = \mathbb{E}Y, \qquad \text{and } 0 \le \frac{d^2\phi}{d\theta^2}(\theta) \le K_2\,, \theta \in \left[0, \frac{\theta_0}{2}\right)$$

*for some* $K_2 > 0$. *As a result of Taylor's expansion, we have*

$$\phi(\theta) \le \theta \mathbb{E}Y + \theta^2 K_2, \quad \text{for } \theta \in [0, 0.5\theta_0).$$

**Lemma 4.7.** *Under the assumption* $b(0,i) = 0, \sigma(0,i) = 0, i \in \mathbb{Z}_+$, *we have*

$$\mathbb{P}_{x,i}\{X(t) = 0 \text{ for some } t \ge 0\} = 0 \text{ for any } x \ne 0, i \in \mathbb{Z}_+.$$

With the auxiliary results above, we can prove our main results.

**Theorem 4.8.** *Suppose that the Markov chain* $\widehat{\alpha}(t)$ *is ergodic with invariant probability measure* $\boldsymbol{\nu} = (\nu_1, \nu_2, \dots)$ *and Assumption 4.3 is satisfied with additional conditions:*

$$\limsup_{i\to\infty} c_i < 0, \tag{4.17}$$

*and*

$$M_g := \sup_{x\in D, i\in\mathbb{Z}_+} \left\{ \left| \frac{V_x(x)\sigma(x,i)}{g(V(x))} \right| \right\} < \infty. \tag{4.18}$$

*Then, if* $\sum_{i\in\mathbb{Z}_+} c_i\nu_i < 0$, *the trivial solution is asymptotic stable in probability, that is, for any* $h > 0$ *such that* $B_h \subset D$, *and* $\varepsilon > 0$, *there exists* $\delta = \delta(h, \varepsilon) > 0$ *such that*

$$\mathbb{P}_{x,i}\left\{X(t) < h \,\forall\, t \ge 0, \quad \text{and} \quad \lim_{t\to\infty} X(t) = 0\right\} > 1 - \varepsilon \text{ for any } (x,i) \in B_\delta \times \mathbb{Z}_+.$$

*Moreover, there is a* $\lambda > 0$ *such that*

$$\mathbb{P}_{x,i}\left\{\lim_{t\to\infty} \frac{V(X(t))}{G^{-1}(-\lambda t)} \le 1\right\} > 1 - \varepsilon \text{ for any } (x,i) \in B_\delta \times \mathbb{Z}_+. \tag{4.19}$$

**Remark 4.9.** Before proceeding to the proof of the theorem, let us make a brief comment. In addition to providing sufficient conditions for asymptotic stability, a significant new element here is the rate of convergence given in (4.19). Although there are numerous treatment of stochastic stability by a host of authors for diffusions and switching diffusions. The rate result in Theorem 4.8 appears to the first one of its kind.

*Proof.* The proof is divided into two steps. We first show the trivial solution is stable in probability and then we prove asymptotic stability and estimate the path-wise convergence rate.

**Step 1: Stability.** Shrinking D if necessary, we can assume without loss of generality that $V(x) \leq 1$ in $D$. Let $h > 0$ such that $B_h \subset D$. Since $\{c_i\}$ is bounded,

$$\lim_{k\to\infty} \sum_{i\leq k} c_i\nu_i = \sum_{i\in\mathbb{Z}_+} c_i\nu_i < 0. \tag{4.20}$$

This and (4.17) show that there exists $k_0 \in \mathbb{Z}_+$ such that

$$-\lambda_1 := \sum_{i\leq k_0} c_i\nu_i < 0$$

and

$$-2\lambda_2 := \sup_{i>k_0} c_i < 0.$$

Let $\bar{c} = \sup_{i\in\mathbb{Z}_+} |c_i|$ and $m_0$ be an positive integer satisfying $m_0\lambda_2 > \bar{c} + M_g + 1$. Define $G(y) = -\int_y^1 g^{-1}(z)dz$. In view of Lemma 4.7, if $X(0) \neq 0$, then $X(t) \neq 0$ a.s, which leads to $g(V(X(t)) \neq 0$ a.s. Thus, we have from Itô's formula and the increasing property of $g(\cdot)$

that

$$
\begin{aligned}
G\big(V(X(\tau_h \wedge t))\big) =& G(V(x)) + \int_0^{\tau_h \wedge t} \frac{\mathcal{L}_{\alpha(s)} V(X(s))}{g(V(X(s)))} ds \\
& - \int_0^{\tau_h \wedge t} \frac{\frac{dg}{dy}(V(X(s)))\Big|V_x(X(s))\sigma(X(s),\alpha(s))\Big|^2}{2g^2(V(X(s)))} ds \\
& + \int_0^{\tau_h \wedge t} \frac{V_x(X(s))\sigma(X(s),\alpha(s))}{g(V(X(s)))} dW(s) \le G(V(x)) + H(t),
\end{aligned}
\tag{4.21}
$$

where

$$
H(t) = \int_0^{\tau_h \wedge t} c_{\alpha(s)} ds + \int_0^{\tau_h \wedge t} \frac{V_x(X(s))\sigma(X(s),\alpha(s))}{g(V(X(s)))} dW(s).
$$

By Itô's formula,

$$
\begin{aligned}
e^{\theta H(t)} =& 1 + \int_0^{t \wedge \tau_h} e^{\theta H(s)} \left[\theta c_{\alpha(s)} + \frac{\theta^2}{2} \frac{|V_x(X(s))\sigma(X(s),\alpha(s))|^2}{2g^2(V(X(s)))}\right] ds \\
& + \theta \int_0^{t \wedge \tau_h} e^{\theta H(s)} \frac{V_x(X(s))\sigma(X(s),\alpha(s))}{g(V(X(s)))} dW(s).
\end{aligned}
\tag{4.22}
$$

Let $\varsigma_k = \inf\{t \ge 0 : |H(t)| \ge k\}$. It follows from (4.22) that

$$
\begin{aligned}
\mathbb{E}_{x,i} e^{\theta H(t \wedge \varsigma_k)} =& 1 + \mathbb{E}_{x,i} \int_0^{t \wedge \varsigma_k \wedge \tau_h} e^{\theta H(s)} \left[\theta c_{\alpha(s)} + \frac{\theta^2}{2} \frac{|V_x(X(s))\sigma(X(s),\alpha(s))|^2}{2g^2(V(X(s)))}\right] ds \\
\le& 1 + [\bar{c} + M_g] \mathbb{E}_{x,i} \int_0^{t \wedge \varsigma_k \wedge \tau_h} e^{\theta H(s)} ds \\
\le& 1 + [\bar{c} + M_g] \int_0^t \mathbb{E}_{x,i} e^{\theta H(s \wedge \varsigma_k)} ds.
\end{aligned}
$$

In view of Gronwall's inequality, for any $t \ge 0$ and $(x, i) \in B_h \times \mathbb{Z}_+$, we have

$$
\mathbb{E}_{x,i} e^{\theta H(t \wedge \varsigma_k)} \le e^{\theta[\bar{c} + M_g]t}, \ \theta \in [-1, 1]. \tag{4.23}
$$

Letting $k \to \infty$ and applying the Lebesgue dominated convergence theorem we obtain

$$\mathbb{E}_{x,i} e^{\theta H(t)} \le e^{\theta[\overline{c}+M_g]t},\ \theta \in [-1,1]. \tag{4.24}$$

On the other hand, we have

$$\begin{aligned}\mathbb{E}_{x,i} H(t) &\le \mathbb{E}_{x,i} \int_0^{\tau_h \wedge t} c_{\alpha(s)} ds \\ &\le \mathbb{E}_{x,i} \int_0^t c_{\alpha(s)} ds - \mathbb{E}_{x,i} \int_{\tau_h \wedge t}^t c_{\alpha(s)} ds \\ &\le \mathbb{E}_{x,i} \int_0^t c_{\alpha(s)} ds + t\overline{c}\mathbb{P}_{x,i}\{\tau_h < t\}.\end{aligned} \tag{4.25}$$

Because of the ergodicity of $\widehat{\alpha}(t)$, there exists a $T > 0$ depending on $k_0$ such that

$$\mathbb{E}_{0,i} \int_0^t c_{\alpha(s)} ds = \mathbb{E}_i \int_0^t c_{\widehat{\alpha}(s)} ds \le -\frac{3\lambda_1}{4} t \quad \forall\, t \ge T, i \le k_0. \tag{4.26}$$

By the Feller property of $(X(t), \alpha(t))$ there exists an $h_1 \in (0, h)$ such that

$$\mathbb{E}_{x,i} \int_0^t c_{\alpha(s)} ds \le -\frac{\lambda_1}{2} t \quad \forall\, t \in [T, T_2], |x| \le h_1, i \le k_0, \tag{4.27}$$

where $T_2 = (m_0 + 1)T$. In view of Lemma 4.5, there exists an $h_2 \in (0, h_1)$ such that

$$\overline{c}\mathbb{P}_{x,i}\{\tau_h < m_0 T + T\} \le \frac{\lambda_1}{4} \text{ provided } |x| \le h_2, i \in \mathbb{Z}_+. \tag{4.28}$$

Applying (4.27) and (4.28) to (4.25), we obtain

$$\mathbb{E}_{x,i} H(t) \le -\frac{\lambda_1}{4} t \text{ if } 0 < |x| \le h_2, i \le k_0, t \in [T, T_2]. \tag{4.29}$$

By Lemma 4.6, it follows from (4.24) and (4.29) that for $\theta \in [0, 0.5], 0 < |x| < h_2, i \le k_0, t \in$

$[T, T_2]$, we have

$$\begin{aligned}\ln \mathbb{E}_{x,i} e^{\theta H(t)} \leq & \theta \mathbb{E}_{x,i} H(t) + \theta^2 K \\ \leq & -\theta \frac{\lambda_1 t}{4} + \theta^2 K\end{aligned} \tag{4.30}$$

for some $K > 0$ depending on $T_2, \overline{c}$ and $M_g$. Let $\theta \in (0, 0.5]$ such that

$$\theta K < \frac{\lambda_1 T}{8}, \text{ and } \theta M_g < \lambda_2 \tag{4.31}$$

we have

$$\ln \mathbb{E}_{x,i} e^{\theta H(t)} \leq -\frac{\theta \lambda_1 t}{8} \text{ for } 0 < |x| < h_2, i \leq k_0, t \in [T, T_2]$$

or equivalently,

$$\mathbb{E}_{x,i} e^{\theta H(t)} \leq \exp\left\{-\frac{\theta \lambda_1 t}{8}\right\} \text{ for } 0 < |x| < h_2, i \leq k_0, t \in [T, T_2]. \tag{4.32}$$

In what follows, we fix a $\theta \in (0, 0.5]$ satisfying (4.31). Exponentiating both sides of the inequality $G(V(X(\tau_h \wedge t))) \leq G(V(x)) + H(t)$ we have for $0 < |x| < h_2, i \leq k_0, t \in [T, T_2]$ that

$$\mathbb{E}_{x,i} U(X(\tau_h \wedge t)) \leq U(x) \mathbb{E}_{x,i} e^{\theta H(t)} \leq U(x) \exp\left\{-\frac{\theta \lambda_1 t}{8}\right\}. \tag{4.33}$$

where $U(x) = \exp(\theta G(V(x)))$. Since $\lim_{x \to 0} G(V(x)) = -\infty$ then

$$\lim_{x \to 0} U(x) = 0. \tag{4.34}$$

Using the inequality $G(V(X(\tau_h \wedge t))) \leq G(V(x)) + H(t)$ and (4.24) we have

$$\mathbb{E}_{x,i} U(X(\tau_h \wedge t)) \leq U(x) \exp\{\theta[\overline{c} + M_g]t\}, \text{ for all } (x, i) \in B_h \times \mathbb{Z}_+, t \geq 0. \tag{4.35}$$

Now, let $\Delta = \inf\{U(x) : h_2 \le |x| \le h\} > 0$. Define stopping times

$$\xi = \inf\{t \ge 0 : \alpha(t) \le k_0\}, \quad \text{and } \zeta = \inf\{t \ge 0 : U(X(t)) \ge \Delta\}.$$

Clearly, if $X(0) \in B_h$ then $\zeta \le \tau_h$ and if $t < \zeta$ then $|X(t)| < h_2$. By computation and (4.31), we have

$$\begin{aligned}\mathcal{L}_i U(x) \le \theta U(x)\left[c_i + [\theta - \dot{g}(V(x))]\frac{|V_x(x)\sigma(x,i)|^2}{g(V(x))}\right] \le& \theta(-2\lambda_2 + \theta M_g)U(x)\\ \le& -\theta\lambda_2 U(x), \text{ for } 0 < |x| < h, i > k_0.\end{aligned}$$

It follows from Itô's formula that

$$\begin{aligned}\mathbb{E}_{x,i} e^{\theta\lambda_2(t\wedge\xi\wedge\zeta)} U(X(t\wedge\xi) =& U(x) + \mathbb{E}_{x,i}\int_0^{t\wedge\xi\wedge\zeta} e^{\lambda_2 s}\left[\theta\lambda_2 U(X(s)) + \mathcal{L}_{\alpha(t)}U(X(s))\right] ds\\ \le& U(x), \text{ for } 0 < |x| < h, i \in \mathbb{Z}_+.\end{aligned} \tag{4.36}$$

We have the following estimate for $0 < |x| < h, i > k_0$.

$$\begin{aligned}\mathbb{E}_{x,i} e^{\theta\lambda_2(T_2\wedge\xi\wedge\zeta)} U(X(T_2\wedge\xi\wedge\zeta)) =& \mathbb{E}_{x,i}\mathbf{1}_{\{\xi\wedge\zeta<m_0T\}} e^{\theta\lambda_2(T_2\wedge\xi\wedge\zeta)} U(X(T_2\wedge\xi\wedge\zeta))\\ &+ \mathbb{E}_{x,i}\mathbf{1}_{\{m_0T\le\xi\wedge\zeta<T_2\}} e^{\theta\lambda_2(T_2\wedge\xi\wedge\zeta)} U(X(T_2\wedge\xi\wedge\zeta))\\ &+ \mathbb{E}_{x,i}\mathbf{1}_{\{\xi\wedge\zeta\ge T_2\}} e^{\theta\lambda_2(T_2\wedge\xi\wedge\zeta)} U(X(T_2\wedge\xi\wedge\zeta))\\ \ge& \mathbb{E}_{x,i}\mathbf{1}_{\{\xi\wedge\zeta\le m_0T\}} U(X(\xi\wedge\zeta))\\ &+ e^{\theta\lambda_2 m_0 T}\mathbb{E}_{x,i}\mathbf{1}_{\{m_0T\le\xi\wedge\zeta<T_2\}} U(X(\xi\wedge\zeta))\\ &+ e^{\theta\lambda_2 T_2}\mathbb{E}_{x,i}\mathbf{1}_{\{\xi\ge T_2\}} U(X(T_2)).\end{aligned} \tag{4.37}$$

Since $\mathbb{P}_{x,i}\{\zeta = 0\} = 1$ if $i \le k_0$, (4.37) holds for $0 < |x| < h, i \in \mathbb{Z}_+$. Noting that $U(x) \wedge \Delta \le \Delta$ for any $x \in B_h$, we have

$$\mathbb{E}\left[U(X(T_2\wedge\tau_h)) \wedge \Delta \middle| \zeta < m_0T, \zeta \le \xi\right] \le \Delta \le U(X(\zeta)) = U(X(\xi\wedge\zeta)).$$

If $\xi < \zeta$, then $U(X(\xi)) < \Delta$. By strong Markov property of $(X(t), \alpha(t))$, (4.33), and (4.24), we have

$$\mathbb{E}\Big[U(X(T_2 \wedge \tau_h)) \wedge \Delta \Big| \xi < m_0 T \wedge \zeta\Big] \le U(X(\xi)) = U(X(\xi \wedge \zeta))$$

and

$$\mathbb{E}\Big[U(X(T_2 \wedge \tau_h)) \wedge \Delta \Big| m_0 T \le \xi < T_2 \wedge \zeta\Big] \le U(X(\xi)) e^{\theta(\bar{c}+M_g)T} = U(X(\xi \wedge \zeta)) e^{\theta(\bar{c}+M_g)T}.$$

From the three estimates above, we have

$$\begin{aligned} \mathbb{E}_{x,i}\mathbf{1}_{\{\xi\wedge\zeta\le m_0T\}}\big[U(X(T_2 \wedge \tau_h)) \wedge \Delta\big] =& \mathbb{E}_{x,i}\mathbf{1}_{\{\zeta< m_0T, \zeta\le\xi\}}\big[U(X(T_2 \wedge \tau_h)) \wedge \Delta\big] \\ &+ \mathbb{E}_{x,i}\mathbf{1}_{\{\xi< m_0T\wedge\zeta\}}\big[U(X(T_2 \wedge \tau_h)) \wedge \Delta\big] \\ \le& \mathbb{E}_{x,i}\mathbf{1}_{\{\xi\wedge\zeta< m_0T\}} U(X(\xi \wedge \zeta)), \end{aligned} \tag{4.38}$$

and,

$$\begin{aligned} \mathbb{E}_{x,i}\mathbf{1}_{\{m_0T\le\xi\wedge\zeta<T_2\}}\big[U(X(T_2 \wedge \tau_h)) \wedge \Delta\big] \le& e^{\theta(\bar{c}+M_g)T}\mathbb{E}_{x,i}\mathbf{1}_{\{m_0T\le\xi\wedge\zeta<T_2\}} U(X(\xi \wedge \zeta)) \\ \le& e^{\theta\lambda_2 m_0 T}\mathbb{E}_{x,i}\mathbf{1}_{\{m_0T\le\xi\wedge\zeta<T_2\}} U(X(\xi \wedge \zeta)), \end{aligned} \tag{4.39}$$

where the last line follows from $m_0\lambda_2 > \bar{c} + M_g + 1$. Applying (4.38) and (4.39) to (4.37), we obtain

$$\mathbb{E}_{x,i}\big[U(X(T_2 \wedge \tau_h)) \wedge \Delta\big] \le U(x) \text{ for any } (x,i) \in B_h \times \mathbb{Z}.$$

Since $\mathbb{E}_{x,i}\big[U(X(T_2 \wedge \tau_h)) \wedge \Delta\big] \le \Delta$, we have

$$\mathbb{E}_{x,i}\big[U(X(T_2 \wedge \tau_h)) \wedge \Delta\big] \le U(x) \wedge \Delta \text{ for any } (x,i) \in B_h \times \mathbb{Z}. \tag{4.40}$$

This together with the Markov property of $(X(t), \alpha(t))$ implies that $\{M(k) := \big[U(X(kT_2 \wedge \tau_h)) \wedge \Delta\big], k \in \mathbb{Z}_+\}$ is a super-martingale. Let $\eta = \inf\{k \in \mathbb{Z}_+ : M(k) = \Delta\}$. Clearly,

$\{\eta < \infty\} \supset \{\tau_h < \infty\}$. For any $\varepsilon > 0$, if $U(x) < \varepsilon\Delta$ we have that

$$\mathbb{P}_{x,i}\{\eta < k\} \leq \frac{\mathbb{E}_{x,i} M(\eta \wedge k)}{\Delta} \leq \frac{U(x)}{\Delta} \leq \varepsilon.$$

Letting $k \to \infty$ yields

$$\mathbb{P}_{x,i}\{\tau_h < \infty\} \leq \mathbb{P}_{x,i}\{\eta < \infty\} \leq \varepsilon, \text{ if } U(x) < \varepsilon\Delta. \tag{4.41}$$

We complete the proof of this step by noting that $\{x : U(x) < \varepsilon\Delta\}$ is a neighborhood of $x$ due to the fact that $\lim_{x\to 0} U(x) = 0$.

**Step 2: Asymptotic stability and pathwise convergence rate.**

To prove the asymptotic stability in probability, we fix $h > 0$ and define $U(x), T_2, m_0, \Delta$ depending on $h$ as in the first step. By virtue of (4.37), we have

$$\begin{aligned}
\mathbb{E}_{x,i} e^{\theta\lambda_2(T_2\wedge\xi\wedge\zeta)} U(X(T_2 \wedge \xi \wedge \zeta)) \geq & \mathbb{E}_{x,i} \mathbf{1}_{\{\xi\wedge\zeta < m_0 T\}} U(X(\xi \wedge \zeta)) \\
& + e^{\theta\lambda_2 m_0 T} \mathbb{E}_{x,i} \mathbf{1}_{\{m_0 T \leq \xi\wedge\zeta < T_2\}} U(X(\xi \wedge \zeta)) \\
& + e^{\theta\lambda_2 T_2} \mathbb{E}_{x,i} \mathbf{1}_{\{\xi\wedge\zeta \geq T_2\}} U(X(T_2)) \\
\geq & \mathbb{E}_{x,i} \mathbf{1}_{\{\xi < m_0 T, \zeta > \xi\}} U(X(\xi)) \\
& + e^{\theta\lambda_2 m_0 T} \mathbb{E}_{x,i} \mathbf{1}_{\{m_0 T \leq \xi < T_2, \zeta > \xi\}} U(X(\xi)) \\
& + e^{\theta\lambda_2 T_2} \mathbb{E}_{x,i} \mathbf{1}_{\{\xi\wedge\zeta \geq T_2\}} U(X(T_2)).
\end{aligned} \tag{4.42}$$

Recalling that $\zeta \le \tau_h$ and $X(t) < h_2$ if $t < \zeta$, we have from (4.32) and (4.35) that

$$\begin{aligned}\mathbb{E}_{x,i}\mathbf{1}_{\{\zeta\ge T_2\}}\mathbf{1}_{\{\xi<m_0T\}}U(X(T_2)) =&\mathbb{E}_{x,i}\mathbf{1}_{\{\zeta\ge T_2\}}\mathbf{1}_{\{\xi<m_0T\}}U(X(T_2\wedge\tau_h))\\ \le&\mathbb{E}_{x,i}\mathbf{1}_{\{\xi<\zeta\}}\mathbf{1}_{\{\xi<m_0T\}}U(X(T_2\wedge\tau_h))\\ \le&\mathbb{E}_{x,i}\left[\mathbf{1}_{\{\xi<m_0T\wedge\zeta\}}U(X(\xi))\exp\left\{-\theta\frac{\lambda}{8}(T_2-\xi)\right\}\right]\\ \le&\exp\left\{-\frac{\theta\lambda T}{8}\right\}\mathbb{E}_{x,i}\left[\mathbf{1}_{\{\zeta\ge\xi\}}\mathbf{1}_{\{\xi<m_0T\}}U(X(\xi))\right]\end{aligned} \tag{4.43}$$

and

$$\begin{aligned}\mathbb{E}_{x,i}\mathbf{1}_{\{m_0T\le\xi<T_2,\zeta\ge T_2\}}U(X(T_2)) \le&\mathbb{E}_{x,i}\mathbf{1}_{\{m_0T\le\xi<T_2\wedge\zeta\}}U(X(T_2\wedge\tau_h))\\ \le&\mathbb{E}_{x,i}\left[\mathbf{1}_{\{m_0T\le\xi<T_2\wedge\zeta\}}U(X(\xi))\exp\left\{\theta(\bar{c}+M_g)(T_2-\xi)\right\}\right]\\ \le&\exp\left\{\theta(\bar{c}+M_g)T\right\}\mathbb{E}_{x,i}\left[\mathbf{1}_{\{m_0T\le\xi<T_2\wedge\zeta\}}U(X(\xi))\right]\\ \le&\exp\{-\theta T\}\exp\left\{\theta\lambda_2m_0T\right\}\mathbb{E}_{x,i}\left[\mathbf{1}_{\{m_0T\le\xi<T_2\wedge\zeta\}}U(X(\xi))\right].\end{aligned} \tag{4.44}$$

On the other hand, we can write

$$\mathbb{E}_{x,i}\mathbf{1}_{\{\xi\wedge\zeta\ge T_2\}}U(X(T_2)) = e^{-\theta\lambda_2T_2}e^{\theta\lambda_2T_2}\mathbb{E}_{x,i}\mathbf{1}_{\{\xi\wedge\zeta\ge T_2\}}U(X(T_2)). \tag{4.45}$$

Letting $p = \max\left\{\exp\left\{-\frac{\theta\lambda T}{8}\right\}, \exp\{-\theta T\}, \exp\{-\theta\lambda_2T_2\}\right\} < 1$ and adding (4.43), (4.44), and (4.44) side by side and then using (4.42) we have

$$\mathbb{E}_{x,i}\mathbf{1}_{\{\zeta\ge T_2\}}U(X(T_2)) \le pU(x),\ \text{ for } (x,i)\in B_h\times\mathbb{Z}_+.$$

By the strong Markov property of the process $(X(t),\alpha(t))$,

$$\begin{aligned}\mathbb{E}_{x,i}\mathbf{1}_{\{\zeta\geq 2T_2\}}U(X(2T_2)) =&\mathbb{E}_{x,i}\left[\mathbf{1}_{\{\zeta\geq T_2\}}\mathbb{E}_{X(T_2),\alpha(T_2)}\mathbf{1}_{\{\zeta\geq T_2\}}U(X(T_2))\right]\\ \leq& p\mathbb{E}_{x,i}\mathbf{1}_{\{\zeta\geq T_2\}}U(X(T_2))\\ \leq& p^2U(x),\ \text{for } (x,i)\in B_h\times\mathbb{Z}_+.\end{aligned}$$

Continuing this way we have

$$\mathbb{E}_{x,i}\mathbf{1}_{\{\zeta\geq kT_2\}}U(X(kT_2))\leq p^kU(x),\ \text{for } (x,i)\in B_h\times\mathbb{Z}_+.$$

Since $2\theta<1$, we have from (4.24) that $\mathbb{E}_{x,i}e^{2\theta H(s)}\leq e^{2\theta[\overline{c}+M_g]s}$. This and the Burkholder-Davis-Gundy inequality imply

$$\begin{aligned}&\mathbb{E}_{x,i}\sup_{t\leq T_2}\left|\int_0^{t\wedge\tau_h}e^{\theta H(s)}\frac{V_x(X(s))\sigma(X(s),\alpha(s))}{g(V(X(s)))}dW(s)\right|\\ &\quad\leq\left[\mathbb{E}_{x,i}\int_0^{T_2\wedge\tau_h}e^{2\theta H(s)}\frac{|V_x(X(s))\sigma(X(s),\alpha(s))|^2}{g^2(V(X(s)))}ds\right]^{\frac{1}{2}}\\ &\quad\leq\left[M_g^2\mathbb{E}_{x,i}\int_0^{T_2}e^{2\theta H(s)}ds\right]^{\frac{1}{2}}\\ &\quad\leq\left[M_g^2\int_0^{T_2}e^{2\theta[\overline{c}+M_g]s}ds\right]^{\frac{1}{2}}:=\widetilde{K}_1.\end{aligned}\tag{4.46}$$

On the other hand

$$\begin{aligned}&\mathbb{E}_{x,i}\sup_{t\leq T_2}\left|\int_0^{t\wedge\tau_h}e^{\theta H(s)}\left[\theta c_{\alpha(s)}+\frac{\theta^2}{2}\frac{|V_x(X(s))\sigma(X(s),\alpha(s))|^2}{2g^2(V(X(s)))}\right]ds\right|\\ &\quad\leq(\overline{c}+M_g)\mathbb{E}_{x,i}\int_0^{T_2\wedge\tau_h}e^{\theta H(s)}ds\\ &\quad\leq(\overline{c}+M_g)\int_0^{T_2}e^{\theta[\overline{c}+M_g]}ds:=\widetilde{K}_2.\end{aligned}\tag{4.47}$$

It follows from (4.46) and (4.47) that

$$\begin{aligned}\mathbb{E}_{x,i}\sup_{t\le T_2}U(X(t\wedge\tau_h)) =&U(x)\mathbb{E}_{x,i}\sup_{t\le T_2}e^{\theta H_t}\\ \le&U(x)[1+\widetilde{K}_1+\widetilde{K}_2]:=U(x)\widetilde{K}_3.\end{aligned}\tag{4.48}$$

By the strong Markov property of $(X(t),\alpha(t))$, we derive from (4.48) that

$$\begin{aligned}&\mathbb{E}_{x,i}\mathbf{1}_{\{\zeta=\infty\}}\sup_{t\in[kT_2,(k+1)T_2]}U(X(t\wedge\tau_h))\\ &\le \mathbb{E}_{x,i}\mathbf{1}_{\{\zeta\ge kT_2\}}\sup_{t\in[kT_2,(k+1)T_2]}U(X(t\wedge\tau_h))\\ &\le \widetilde{K}_3\mathbb{E}_{x,i}\mathbf{1}_{\{\zeta\ge kT_2\}}U(X(kT_2)\\ &\le \widetilde{K}_3U(x)\rho^k,\end{aligned}\tag{4.49}$$

which combined with Markov's inequality leads to

$$\begin{aligned}&\mathbb{P}_{x,i}\left\{\mathbf{1}_{\{\zeta=\infty\}}\sup_{t\in[kT_2,(k+1)T_2]}U(X(t\wedge\tau_h))>(\rho+\widetilde{\varepsilon})^k\right\}\\ &\le\frac{1}{(\rho+\widetilde{\varepsilon})^k}\mathbb{E}_{x,i}\left[\mathbf{1}_{\{\zeta=\infty\}}\sup_{t\in[kT_2,(k+1)T_2]}U(X(t\wedge\tau_h))\right]\\ &\le\widetilde{K}_3U(x)\frac{\rho^k}{(\rho+\widetilde{\varepsilon})^k}\quad k\in\mathbb{Z}_+,\end{aligned}\tag{4.50}$$

where $\widetilde{\varepsilon}$ is any number in $(0,1-\rho)$. In view of the Borel-Cantelli lemma, for almost all $\omega\in\Omega$, there exists random integer $k_1=k_1(\omega)$ such that

$$\mathbf{1}_{\{\zeta=\infty\}}\sup_{t\in[kT_2,(k+1)T_2]}U(X(t))<(\rho+\widetilde{\varepsilon})^k\ \text{ for any } k\ge k_1.$$

Thus, for almost all $\omega\in\{\zeta=\infty\}$, we have

$$G(V(X(t)))\le[t/T_2]\ln(\rho+\widetilde{\varepsilon})\le-\lambda t\ \text{ for } t\ge k_1T_2.\tag{4.51}$$

where $[t/T_2]$ is the integer part of $t/T_2$ and $\lambda=-\dfrac{\ln(\rho+\widetilde{\varepsilon})}{2T_2}>0$. Since $G(y)$ is decreasing and maps $(0, h]$ onto $(-\infty, 0]$, (4.19) follows from (4.41) and (4.51). The proof is complete.

□

In Theorem 4.8, under the condition that $\alpha(t)$ is merely ergodic, we need and additional condition (4.17) to obtain the stability in probability of the system. If $\alpha(t)$ is strongly ergodic, the condition (4.17) can be removed.

**Theorem 4.10.** *Suppose that*

- *for any $T > 0$ and a bounded function $f : \mathbb{Z}_+ \mapsto \mathbb{R}$, we have*

$$\lim_{x \to 0} \sup_{i \in \mathbb{Z}_+} \left\{ \left| \mathbb{E}_{x,i} \int_0^T f(\alpha(s))ds - \mathbb{E}_i \int_0^T f(\widehat{\alpha}(s)ds \right| \right\} = 0. \tag{4.52}$$

- *Assumption 4.3 is satisfied*
- *the Markov chain $\widehat{\alpha}(t)$ is strongly ergodic with invariant probability measure $\boldsymbol{\nu} = (\nu_1, \nu_2, \dots)$.*

*Suppose further that (4.18) is satisfied and $\sum_{i \in \mathbb{Z}_+} c_i \nu_i < 0$. Then the conclusion of Theorem 4.8 holds.*

**Remark 4.11.** We will prove in the Appendix that (4.52) holds if Assumption 4.3 and (4.8) hold.

*Proof of Theorem 4.10.* Let $\lambda = -\sum_{i \in \mathbb{Z}_+} c_i \nu_i$. Because of the uniform ergodicity of $\widehat{\alpha}(t)$, there exists a $T > 0$ such that

$$\mathbb{E}_{0,i} \int_0^t c_{\alpha(s)} ds = \mathbb{E}_i \int_0^t c(\widehat{\alpha}(s))ds \le -\frac{3\lambda}{4} t \quad \forall\, t \ge T, i \in \mathbb{Z}_+. \tag{4.53}$$

By (4.52), there exists an $h_1 \in (0, h)$ such that

$$\mathbb{E}_{x,i} \int_0^T c_{\alpha(s)} ds \le -\frac{\lambda}{2} T \quad \forall\, |x| \le h_1, i \in \mathbb{Z}_+. \tag{4.54}$$

In view of Lemma 4.5, there exists an $h_2 \in (0, h_1)$ such that

$$\bar{c}\mathbb{P}_{x,i}\{\tau_h < T\} \leq \frac{\lambda}{4} \text{ provided } |x| \leq h_2, i \in \mathbb{Z}_+. \tag{4.55}$$

Applying (4.54) and (4.55) to (4.25), we have

$$\mathbb{E}_{x,i} H(T) \leq -\frac{\lambda}{4} T \text{ if } 0 < |x| \leq h_2, i \in \mathbb{Z}_+. \tag{4.56}$$

Using (4.56), we can use arguments in the proof Theorem 4.8 to show that

$$\mathbb{E}_{x,i} e^{\theta H(T)} \leq \exp\left\{-\frac{\theta \lambda T}{8}\right\} \text{ for } 0 < |x| < h_2 \tag{4.57}$$

for a sufficiently small $\theta > 0$. This implies that

$$\mathbb{E}_{x,i} U\big(X(T \wedge \tau_h)\big) \leq \exp\left\{-\frac{\theta \lambda T}{8}\right\} U(x), \tag{4.58}$$

where $U(x) = \exp(\theta G(V(x)))$. Thus, $\{M_k := U\big(X\big((kT) \wedge \tau_h\big)\big), k = 0, 1, \dots\}$ is a bounded supermartingale. Then we can easily obtain the stability in probability of the trivial solution. Moreover, proceeding as in Step 2 of the proof of Theorem 4.8, we can obtain the asymptotic stability as well as the rate of convergence. The arguments are actually simpler because (4.58) holds uniformly in $i \in \mathbb{Z}_+$, rather than $i \in \{1, \dots, k_0\}$ in the proof of Theorem 4.8. □

**Remark 4.12.** Consider the special case $g(y) \equiv y$. With this function we have $U(X(t)) = V(X(t))$. Thus, if Assumption 4.3 holds with $g(y) \equiv y$, then the conclusion on stability in Theorems 4.8 and 4.10 are still true without the condition (4.18) because we still have $\mathbb{E}V(X(t \wedge \tau_h)) \leq V(x) e^{\bar{c}t}$, which can be used in place of (4.35). However, in order to obtain asymptotic stability and rate of convergence, (4.18) is needed. In that case, if the initial value is sufficiently closed to 0, $V(X(t))$ will converges exponentially fast to 0 with a large probability.

**Theorem 4.13.** *Consider the case that the state space of $\alpha(t)$ is finite, say $\mathcal{M} = \{1, \dots, m_0\}$ for some positive integer $m_0$, rather than $\mathbb{Z}_+$. Suppose that $Q(0)$ is irreducible and let $\boldsymbol{\nu}$ be the invariant probability measure of the Markov chain with generator $Q(0)$. If $\sum_{i\in\mathcal{M}} c_i\nu_i < 0$ then the trivial solution is asymptotically stable in probability, and for any $\varepsilon > 0$, there are $\lambda > 0, \delta > 0$ such that*

$$\mathbb{P}_{x,i}\left\{\lim_{t\to\infty} \frac{V(X(t))}{G^{-1}(-\lambda_3 t)} \leq 1\right\} > 1 - \varepsilon \text{ for any } (x,i) \in B_\delta \times \mathcal{M}.$$

We now provide some conditions for instability in probability.

**Theorem 4.14.** *Suppose that the Markov chain $\widehat{\alpha}(t)$ is ergodic with invariant probability measure $\boldsymbol{\nu} = (\nu_1, \nu_2, \dots)$ and that there are functions $g \in \Gamma$, $V : D \mapsto \mathbb{R}_+$ such that*

- *$V(x) = 0$ if and only if $x = 0$*
- *$V(x)$ is continuous on $D$ and twice continuously differentiable in $D \setminus \{0\}$.*
- *there is a bounded sequence of real numbers $\{c_i : i \in \mathbb{Z}_+\}$ such that*

$$\mathcal{L}_i V(x) \geq c_i g(V(x))\, \forall\, x \in D \setminus \{0\}. \tag{4.59}$$

*If (4.18) is satisfied and if $\sum_{i\in\mathbb{Z}_+} c_i\nu_i < 0$ and $\limsup_{i\to\infty} c_i < 0$, then the trivial solution is unstable in probability.*

*Proof.* Define $G(y) = -\int_y^1 g^{-1}(z)dz$ as in Theorem 4.8. We have from Itô's formula,

$$\begin{aligned} -G\big(V(X(\tau_h \wedge t))\big) = &- G(V(x)) - \int_0^{\tau_h\wedge t} \frac{\mathcal{L}_{\alpha(s)} V(X(s))}{g(V(X(s)))} ds \\ &+ \int_0^{\tau_h\wedge t} \frac{\dot{g}(V(X(s)))|V_x(X(s))\sigma(X(s),\alpha(s))|^2}{2g^2(V(X(s)))} ds \\ &- \int_0^{\tau_h\wedge t} \frac{V_x(X(s))\sigma(X(s),\alpha(s))}{g(V(X(s)))} dW(s) \leq -G(V(x)) + \widetilde{H}(t) \end{aligned} \tag{4.60}$$

where

$$\widetilde{H}(t) = -\int_0^{\tau_h\wedge t} c_{\alpha(s)}ds - \int_0^{\tau_h\wedge t} \frac{V_x(X(s))\sigma(X(s),\alpha(s))}{g(V(X(s)))}dW(s).$$

Then using (4.60) and proceeding in the same manner as in the proof of Theorem 4.8 with $H(t)$ replaced with $\widetilde{H}(t)$, we have can find a sufficiently small $\widetilde{\theta}, \widetilde{\Delta} > 0$ and a sufficiently large $T_3 > 0$ such that

$$\mathbb{E}_{x,i}\mathbf{1}_{\{\widetilde{\zeta}\geq kT_3\}}\widetilde{U}(X(kT_2)) \leq p^k\widetilde{U}(x), \text{ for } (x,i) \in B_h \times \mathbb{Z}_+.$$

where $\widetilde{U}(x) = \exp\big\{-\widetilde{\theta}G(V(x))\big\}$, and $\widetilde{\zeta} = \inf\{k \geq 0 : U(X(kT_3)) \leq \widetilde{\Delta}^{-1}\}$. Note that, unlike $U(x)$, we have $\lim_{x\to 0}\widetilde{U}(x) = \infty$. Since $U(X(kT_3)) \geq \widetilde{\Delta}^{-1}$ if $\widetilde{\zeta} \geq k$, we have that

$$\mathbb{P}_{x,i}\{\widetilde{\zeta} = \infty\} = \lim_{k\to\infty}\mathbb{P}_{x,i}\{\widetilde{\zeta} \geq k\} = 0.$$

□

Similarly, we can obtain a counterpart of Theorem 4.10 for instability.

**Theorem 4.15.** *Suppose that the Markov chain $\widehat{\alpha}(t)$ is strongly ergodic with invariant probability measure $\boldsymbol{\nu} = (\nu_1, \nu_2, \dots)$ and that there are functions $g \in \Gamma$, $V : D \mapsto \mathbb{R}_+$ such that*

- *$V(x) = 0$ if and only if $x = 0$*
- *$V(x)$ is continuous on $D$ and twice continuously differentiable in $D \setminus \{0\}$.*
- *there is a bounded sequence of real numbers $\{c_i : i \in \mathbb{Z}_+\}$ such that*

$$\mathcal{L}_iV(x) \geq c_ig(V(x))\,\forall\, x \in D \setminus \{0\}. \tag{4.61}$$

*If (4.18) and (4.8) are satisfied and if $\sum_{i\in\mathbb{Z}_+} c_i\nu_i > 0$ then the trivial solution is unstable in probability.*

## 4.3 Stability and Instability of Linearized Systems

Suppose that (4.52) is satisfied and that $\widehat{\alpha}(t)$ is a strongly ergodic Markov chain.

**Assumption 4.4.** *Suppose that for $i \in \mathbb{Z}_+$, there exist $b(i), \sigma_k(i) \in \mathbb{R}^{n\times n}$ bounded uniformly for $i \in \mathbb{Z}_+$ such that*

$$\xi_i(x) := b(x,i) - b(i)x, \quad \zeta_i(x) := \sigma(x,i) - (\sigma_1(i)x, \dots, \sigma_d(i)x)$$

*satisfying*

$$\lim_{x\to 0} \sup_{i\in\mathbb{Z}_+} \left\{ \frac{|\xi_i(x)| \vee |\zeta_i(x)|}{|x|} \right\} = 0. \tag{4.62}$$

For $i \in \mathbb{Z}_+$, $k \in \{1, \dots, n\}$, let $\Lambda_{1,i}$ and $\Lambda_{2,i,k}$ be the maximum eigenvalues of $\dfrac{b(i) + b^\top(i)}{2}$ and $\sigma_k(i)\sigma_k^\top(i)$ respectively. Similarly, denote by $\lambda_{1,i}$ and $\lambda_{2,i,k}$ be the minimum eigenvalues of $\dfrac{b(i) + b^\top(i)}{2}$ and $\sigma_k(i)\sigma_k^\top(i)$ respectively.

Suppose that $\Lambda_{1,i}$ and $\Lambda_{2,i,k}$ are bounded in $i \in \mathbb{Z}_+$ then we claim that if

$$\sum_{i\in\mathbb{Z}_+} \nu_i \left( \Lambda_{1,i} + \frac{1}{2} \sum_{k=1}^n \Lambda_{2,i,k} \right) < 0,$$

then the trivial solution is asymptotic stable.

To show that, let $\varepsilon > 0$ be sufficiently small such that

$$\sum_{i\in\mathbb{Z}_+} \nu_i \left( \varepsilon + \Lambda_{1,i} + \frac{1}{2} \sum_{k=1}^n \Lambda_{2,i,k} \right) < 0. \tag{4.63}$$

Define $V(x) = |x|^p$, carry out the calculation and obtain the estimates as in that of [26, Theorem 4.3], we can find a sufficiently small $p > 0$ and $\hbar > 0$ such that

$$\mathcal{L}_i V(x) \le p \left( \varepsilon + \Lambda_{1,i} + \frac{1}{2} \sum_{k=1}^n \Lambda_{2,i,k} \right) V(x) \text{ for } 0 < |x| < \hbar. \tag{4.64}$$

(Note that the existence of such $p$ and $\hbar$ satisfying (4.64) uniformly for $i \in \mathbb{Z}_+$ is due to (4.64) and the boundedness of $\Lambda_{1,i}$ and $\Lambda_{2,i,k}$.)

By (4.63) and (4.64), it follows from Theorem 4.10 that the trivial solution is asymptotic stable and for any $\varepsilon > 0$, there exists $\delta > 0$, $\lambda > 0$ such that

$$\mathbb{P}_{x,i}\left\{\lim_{t\to\infty} e^{\lambda t}|X(t)| \leq 1\right\} \geq 1-\varepsilon \text{ for } (x,i) \in B_\delta \times \mathbb{Z}_+.$$

Similarly, if $\sum_{i\in\mathbb{Z}_+} \nu_i \left(\lambda_{1,i} + \frac{1}{2}\sum_{k=1}^n \lambda_{2,i,k}\right) > 0$, and if $\lambda_{1,i}$ and $\lambda_{2,i,k}$ are bounded in $i \in \mathbb{Z}_+, k = 1, \ldots, n$, we have that the trivial solution is unstable. To sum up, we have the following result.

**Proposition 4.16.** *Let Assumption 4.4 is satisfied. We claim that,*

- *if $\Lambda_{1,i}$ and $\Lambda_{2,i,k}$ are bounded in $i \in \mathbb{Z}_+$ and*

$$\sum_{i\in\mathbb{Z}_+} \nu_i \left(\Lambda_{1,i} + \frac{1}{2}\sum_{k=1}^n \Lambda_{2,i,k}\right) < 0,$$

  *then the trivial solution is asymptotically stable in probability;*

- *if $\lambda_{1,i}$ and $\lambda_{2,i,k}$ are bounded in $i \in \mathbb{Z}_+, k = 1, \ldots, n$, and*

$$\sum_{i\in\mathbb{Z}_+} \nu_i \left(\lambda_{1,i} + \frac{1}{2}\sum_{k=1}^n \lambda_{2,i,k}\right) > 0,$$

  *then the trivial solution is unstable in probability.*

## 4.4 Examples

This section provides several examples.

**Example 4.17.** *Consider a real-valued switching diffusion*

$$dX(t) = b(\alpha(t))X(t)[|X(t)|^\gamma \vee 1]dt + \sigma(\alpha(t))\sin^2 X(t)dW(t),\ 0 < \gamma < 1, \tag{4.65}$$

*where* $a \vee b = \max(a,b)$ *for two real numbers* $a$ *and* $b$*, and* $Q(x) = \big(q_{ij}(x)\big)_{\mathbb{Z}_+\times\mathbb{Z}_+}$ *with*

$$q_{ij}(x) = \begin{cases} -\check{p}_1(x) & \text{if } i = j = 1 \\ \check{p}_1(x) & \text{if } i = 1, j = 2 \\ -\widehat{p}_i(x) - \check{p}_i(x) & \text{if } i = j \geq 2 \\ \widehat{p}_i(x) & \text{if } i \geq 2, j = i-1 \\ \check{p}_i(x) & \text{if } i \geq 2, j = i+1. \end{cases}$$

*Note that the drift grow faster than linear and the diffusion coefficient is locally like* $x^2$ *near the origin for the continuous state. Suppose that* $b(i), \sigma(i), \check{p}_i(x), \widehat{p}_i(x)$ *are bounded for* $(x,i) \in \mathbb{R}\times\mathbb{Z}_+$ *and* $\check{p}_i(x), \widehat{p}_i(x)$ *are continuous in* $\mathbb{R}^n$ *for each* $i \in \mathbb{Z}_+$*. It is well known (see [1, Chapter 8]) that if*

$$\nu^* := \sum_{k=2}^{\infty}\prod_{\ell=2}^{k}\frac{\check{p}_{\ell-1}(0)}{\widehat{p}_\ell(0)} < \infty,$$

*then* $\widehat{\alpha}(t)$ *is ergodic with the invariant measure* $\boldsymbol{\nu}$ *given by*

$$\nu_1 = \frac{1}{\nu^*},\ \nu_k = \frac{1}{\nu^*}\prod_{\ell=2}^{k}\frac{\check{p}_{\ell-1}(0)}{\widehat{p}_\ell(0)}, k \geq 2.$$

*We suppose that*

$$\sum b(i)\nu_i < 0, \quad \textit{and} \ \limsup_{i\to\infty} b(i) < 0.$$

*we will show that the trivial solution is stable. Let* $0 < \varepsilon < -\sum b(i)\nu_i$ *then* $\sum[b(i)+\varepsilon]\nu_i < 0$*. Let*

$$V(x) = x^2$$

*We have*

$$\mathcal{L}_i V(x) = 2b(i)|x|^{2+2\gamma} + \sigma^2(i)\sin^4(x)$$

*Since* $\gamma < 1$ *and* $\sigma(i)$ *is bounded, there exists an* $\hbar > 0$ *such that* $\sigma^2(i)\sin^4(x) \le \varepsilon|x|^{2+2\gamma}$ *given that* $|x| \le \hbar$. *then*

$$\mathcal{L}_i V(x) \le [2b(i)+\varepsilon]|x|^{2+2\gamma} = [2b(i)+\varepsilon]V^{1+\gamma}(x) \ \ \text{in } [-\hbar,\hbar]\times\mathbb{Z}_+.$$

*By Theorem 4.8, the trivial solution is asymptotically stable in probability. Moreover, for the function* $g(y) = y^{1+\gamma}$,

$$G(y) := -\int_y^1 \frac{1}{g(z)}ds = 1 - y^{-\gamma},\ y \in (0,1]$$

*has the inverse*

$$G^{-1}(-t) = \frac{1}{[t+1]^{1/\gamma}},\ \ \text{for } t \ge 0$$

*Thus, for any* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that if* $(x,i) \in [0,\delta]\times\mathbb{Z}_+$, *then, there exists a* $\lambda > 0$ *such that*

$$\mathbb{P}_{x,i}\left\{\limsup_{t\to\infty} t^{1/\gamma}X^2(t) \le \lambda\right\} > 1-\varepsilon.$$

**Example 4.18.** *This example consider a random-switching linear systems of differential equations:*

$$dX(t) = A(\alpha(t))X(t)dt \tag{4.66}$$

*where* $A(i) \in \mathbb{R}^{n\times n}$ *satisfying* $\sup_{i\in\mathbb{Z}_+}\{|\lambda_i| \vee |\Lambda_i|\} < \infty$ *with* $\lambda_i, \Lambda_i$ *being the minimum and* *maximum eigenvalues of* $A(i)$*, respectively. Let*

$$Q(x) = \begin{pmatrix} -1-\sin|x| & 1+\sin|x| & 0 & 0 & 0 & \cdots \\ 1+\sin|x| & -2-2\sin|x| & 1+\sin|x| & 0 & 0 & \cdots \\ 1+\sin|x| & 0 & -2-2\sin|x| & 1+\sin|x| & 0 & \cdots \\ 1+\sin|x| & 0 & 0 & -2-2\sin|x| & 1+\sin|x| & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

*By [1, Proposition 3.3], it is easy to verify that the Markov chain* $\widehat{\alpha}(t)$ *with generators* $Q(0)$ *is strongly ergodic. Solving the system*

$$\boldsymbol{\nu} Q(0) = 0, \sum \boldsymbol{\nu}_i = 1$$

*we obtain that the invariant measure of* $\widehat{\alpha}(t)$ *is* $(\nu_i)_{i=1}^{\infty} = (2^{-i})_{i=1}^{\infty}$*. Thus, if* $\sum \lambda_i 2^{-i} > 0$ *the trivial solution to* (4.66) *is unstable. In case* $\sum \Lambda_i 2^{-i} < 0$ *the trivial solution to* (4.66) *is asymptotically stable in probability. In particular, suppose that* $n = 2$ *and* $A(i)$ *are upper triangle matrices, that is,*

$$A(i) = \begin{pmatrix} a_i & b_i \\ 0 & c_i \end{pmatrix}.$$

*If* $a_i$ *and* $c_i$ *are positive for* $i \geq 2$*, then the system* $dX(t) = A(i)X(t)dt$ *is unstable. However, if* $a_1, c_1 < -\sup_{i\geq 2}\{a_i, c_i\}$*, then* $\sum (a_i \vee c_i) 2^{-i} < 0$*. Thus, the switching differential system is asymptotically stable. The stability of the system at state 1 and the switching process become a stabilizing factor.*

*On the other hand, if* $a_i \wedge c_i$ *is negative for* $i \geq 2$*, then the system* $dX(t) = A(i)X(t)dt$ *is asymptotically stable. Suppose further that* $a_1, c_1 > \sup_{i\geq 2}\{-(a_i \wedge c_i)\}$*, then* $\sum (a_i \vee c_i) 2^{-i} > 0$*.*

*Under this condition, the switching differential system is unstable.*

## 4.5 Further Remarks

Using a new method, we provide sufficient conditions for stability and instability in probability of a class of regime-switching diffusion systems with switching states belonging to a countable set. The conditions are based on the relation of a "switching-independent" Lyapunov function and the generator of the switching part.

Although the systems under consideration are memoryless, the main results of this paper hold if we assume that the switching intensities $q_{ij}$ depend on the history of $\{X(t)\}$ rather than the current state of $X(t)$, (see [34, 36] for fundamental properties of this process). The problem can be formulated as follows. Let $r$ be a fixed positive number. Denote by $\mathcal{C}$ the set of $\mathbb{R}^n$-valued continuous functions defined on $[-r, 0]$. For $\phi \in \mathcal{C}$, we use the norm $\|\phi\| = \sup\{|\phi(t)| : t \in [-r, 0]\}$. For $t \geq 0$, we denote by $y_t$ the so-called segment function (or memory segment function) $y_t = \{y(t+s) : -r \leq s \leq 0\}$. We assume that the jump intensity of $\alpha(t)$ depends on the trajectory of $X(t)$ in the interval $[t-r, t]$. That is, there are functions $q_{ij}(\cdot) : \mathcal{C} \to \mathbb{R}$ for $i, j \in \mathbb{Z}_+$ satisfying that $q_i(\phi) := \sum_{j=1, j\neq i}^{\infty} q_{ij}(\phi)$ is uniformly bounded in $(\phi, i) \in \mathcal{C} \times \mathbb{Z}_+$ and that $q_i(\cdot)$ and $q_{ij}(\cdot)$ are continuous such that

$$\begin{aligned} &\mathbb{P}\{\alpha(t+\Delta) = j | \alpha(t) = i, X_s, \alpha(s), s \leq t\} = q_{ij}(X_t)\Delta + o(\Delta) \text{ if } i \neq j \text{ and} \\ &\mathbb{P}\{\alpha(t+\Delta) = i | \alpha(t) = i, X_s, \alpha(s), s \leq t\} = 1 - q_i(X_t)\Delta + o(\Delta). \end{aligned} \tag{4.67}$$

It is proved in Chapter 2 that if either Assumption 4.4 or Assumption 4.2 is satisfied with $x, y \in \mathbb{R}^n$ replaced by $\phi, \psi \in \mathcal{C}$, then there is a unique solution to the switching diffusion (4.1) and (4.67) with a given initial value. Moreover, the process $(X_t, \alpha(t))$ has the Markov-

Feller property. With slight modification in the proofs, the theorems in Section 3 still hold for system (4.1) and (4.67).

Our method can also be applied to regime-switching jump diffusion processes. The results obtain by using our method will generalize existing results (e.g., [56, 59]) to the case of regime-switching jump diffusions with countable regimes.

On the other hand, there is a gap between sufficient conditions for stability and instability in Proposition 4.1. To overcome the difficulty, we need to make a polar coordinate transformation to decompose of $X(t)$ into the radial part $r(t) = |X(t)|$ and the angular part $Y(t) = X(t)/r(t)$. Then, the Lyapunov exponents with respect to invariant measures of the linearized process of $(Y(t), \alpha(t))$ will determine whether or not the system is stable. This approach has been used to treat many linear and linearized stochastic systems (e.g., [2, 5, 24]). In our setting, the switching $\alpha(t)$ take values in a noncompact space, thus, it is more difficult to examine invariant measures. We will address this problem together necessary conditions of stability in a subsequent paper.

# APPENDIX A: SUPPLEMENTS FOR CHAPTER 2

This section is devoted to the proofs of some technical results. To simplify the notation, we denote by $\mathbb{P}_{\phi,i}$ the probability measure conditional on the initial value $(\phi, i)$, that is, for any $t > 0$,

$$\mathbb{P}_{\phi,i}\{(X_t, \alpha(t)) \in \cdot\} = \mathbb{P}\{(X_t^{\phi,i}, \alpha^{\phi,i}(t)) \in \cdot\},$$

$$\mathbb{P}_{\phi,i}\{(Y_t, \beta(t)) \in \cdot\} = \mathbb{P}\{(Y_t^{\phi,i}, \beta^{\phi,i}(t)) \in \cdot\},$$

and

$$\mathbb{P}_{\phi,i}\{(Z_t, \gamma(t)) \in \cdot\} = \mathbb{P}\{(Z_t^{\phi,i}, \gamma^{\phi,i}(t)) \in \cdot\}.$$

Let $\mathbb{E}_{\phi,i}$ be the expectation associated with $\mathbb{P}_{\phi,i}$. First, we prove the following result.

**Lemma A.1.** *Let either Assumption* 2.1 *or Assumption* 2.3 *combined with* (ii) *of Assumption* 2.1 *be satisfied. Assume further that $q_i(\cdot), q_{ij}(\cdot)$ are continuous functions in $\mathcal{C}$ for each $i, j \in \mathbb{Z}_+$. Then the solution $(X_t, \alpha(t))$ to* (1.7) *satisfies* (1.5) *and* (1.6).

*Proof.* It is clear that the solution $(X_t, \alpha(t))$ to (1.7) satisfies (1.5). Fix $\phi \in \mathcal{C}$, $i, j \in \mathbb{Z}_+, i \neq j$. Applying the generalized Itô formula to the function $V(\psi, k) = 0$ if $k \neq j$ and $V(\psi, j) = 1$ we have

$$\mathbb{P}_{\phi,i}\{\alpha(\Delta) = j\} = \mathbb{E}_{\phi,i} V(X_\Delta, \alpha(\Delta) = \mathbb{E}_{\phi,i} \int_0^\Delta q_{\alpha(t),j}(X_t)dt, \quad \text{for } \Delta > 0.$$

where $q_{ii}(\phi) := -q_i(\phi) = -\sum_{j\neq i} q_{ij}(\phi)$. Since $\alpha(t)$ is cadlag and $X(t)$ is continuous, $\lim_{t\to 0^+} \alpha(t) = i$ and $\lim_{t\to 0^+} X_t = \phi$ $\mathbb{P}_{\phi,i}$-a.s. In light of the continuity of $q_{ij}(\cdot)$ we obtain

$\lim_{t\to 0^+} q_{\alpha(t),j}(X_t) = q_{ij}(\phi)\ \mathbb{P}_{\phi,i} -$ a.s. which implies that

$$\lim_{\Delta\to 0^+} \frac{1}{\Delta}\int_0^\Delta q_{\alpha(t),j}(X_t)dt = q_{ij}(\phi)\ \mathbb{P}_{\phi,i} - \text{a.s.}$$

Since $q_{ij}(\cdot)$ is uniformly bounded, so is $\frac{1}{\Delta}\int_0^\Delta q_{\alpha(t),j}(X_t)dt$. By virtue of the Lebegue dominated convergence theorem, we have

$$\lim_{\Delta\to 0^+} \frac{\mathbb{P}_{\phi,i}\{\alpha(\Delta)=j\}}{\delta} = \lim_{\Delta\to 0^+} \mathbb{E}_{\phi,i}\left(\frac{1}{\Delta}\int_0^\Delta q_{\alpha(t),j}(X_t)dt\right) = q_{ij}(\phi). \tag{A.1}$$

In the same manner, applying the generalized Itô formula to the function $V(\psi,k) = 1$ if $k \neq i$ and $V(\psi,i) = 0$, we obtain that

$$\lim_{\Delta\to 0^+} \frac{1-\mathbb{P}_{\phi,i}\{\alpha(\Delta)=i\}}{\delta} = q_i(\phi). \tag{A.2}$$

The proof is complete by noting that (1.6) follows from (A.1) and (A.2) and the Markov property of $(X(t),\alpha(t))$. □

Next, we provide the proofs of some results in Section 4.

*Proof of Lemma 2.9.* To prove claim (i), we apply the generalized Itô formula to $V(j) = 1$ if $j = i$, and $V(j) = 0$ if $j \neq i$. We have

$$V(\beta(\lambda_1\wedge t)) = -\int_0^{\lambda_1\wedge t} q_i(Y_s)ds + \int_0^{\lambda_1\wedge t}\int_{\mathbb{R}} \Big(V\big(i+h(Y_t,i,z)\big)-1\Big)\mu(ds,dz).$$

Since $W(\cdot)$ is independent of the Poisson random measure, taking the conditional expectation with respect to $\mathcal{F}_T^W$ yields

$$\begin{aligned}\mathbb{E}_{\phi,i}\big[\mathbf{1}_{\{\lambda_1>t\}}\big|\mathcal{F}_T^W\big] =&\mathbb{E}_{\phi,i}\big[V(\lambda_1\wedge t)\big|\mathcal{F}_T^W\big] = -\mathbb{E}_{\phi,i}\Big[\int_0^{\lambda_1\wedge t} q_i(Y_s)ds\big|\mathcal{F}_T^W\big] + 1\\ =& -\mathbb{E}_{\phi,i}\Big[\int_0^t q_i(Y_s)ds\mathbf{1}_{\{\lambda_1>s\}}\big|\mathcal{F}_T^W\big] + 1\\ =& -\int_0^t q_i(Y_s)\mathbb{E}_{\phi,i}\big[\mathbf{1}_{\{\lambda_1>s\}}\big|\mathcal{F}_T^W\big]ds + 1.\end{aligned}$$

Hence,

$$\frac{d}{dt}\mathbb{E}_{\phi,i}\big[\mathbf{1}_{\{\lambda_1>t\}}\big|\mathcal{F}_T^W\big] = -q_i(Y_t)\mathbb{E}_{\phi,i}\big[\mathbf{1}_{\{\lambda_1>t\}}\big|\mathcal{F}_T^W\big].$$

Since $\mathbb{E}_{\phi,i}\big[\mathbf{1}_{\{\lambda_1>0\}}\big|\mathcal{F}_T^W\big] = 1$, we obtain

$$\mathbb{P}_{\phi,i}\big(\{\lambda_1>t\}\big|\mathcal{F}_T^W\big) = \mathbb{E}_{\phi,i}\big[\mathbf{1}_{\{\lambda_1>t\}}\big|\mathcal{F}_T^W\big] = \exp\Big(-\int_0^t q_i(Y_s)ds\Big). \tag{A.3}$$

Now we prove claim (ii). First, we try to find the distribution of $(\lambda_1, \beta_1)$ conditioned on $\mathcal{F}_T^W$ when $\lambda_1 \in [0, T]$. Fix $j \neq i$ and let $f(t, k) : [0, T] \times \mathbb{Z}_+ \to \mathbb{Z}_+$ be any bounded measurable function satisfying $f(t, k) = 0$ if $k \neq j$. Applying the generalized Itô formula, we obtain

$$\begin{aligned}f(\lambda_1\wedge T, \beta(\lambda_1\wedge T)) =& \int_0^{\lambda_1\wedge T} q_{ij}(Y_t)f(t,j)dt\\ &+ \int_0^{\lambda_1\wedge T}\int_{\mathbb{R}}\Big(f\big(s, i+h(Y_t,i,z)\big) - f(Y_t,i)\Big)\mu(ds,dz).\end{aligned}$$

Since $W(\cdot)$ is independent of the Poisson random measure, taking the conditional expectation with respect to $\mathcal{F}_T^W$, we have

$$\begin{aligned}\mathbb{E}_{\phi,i}\big[f(\lambda_1\wedge T,\beta(\lambda_1\wedge T))\big|\mathcal{F}_T^W\big] =&\mathbb{E}_{\phi,i}\Big[\int_0^{\lambda_1\wedge T} q_{ij}(Y_t)f(t,j)dt\Big|\mathcal{F}_T^W\Big]\\ =&\mathbb{E}_{\phi,i}\Big[\int_0^{T} q_{ij}(Y_t)f(t,j)dt\mathbf{1}_{\{\lambda_1>t\}}dt\Big|\mathcal{F}_T^W\Big]\\ =&\int_0^T q_{ij}(Y_t)f(t,j)\mathbb{E}\big[\mathbf{1}_{\{\lambda_1>t\}}\big|\mathcal{F}_T^W\big]dt\\ =&\int_0^T q_{ij}(Y_t)f(t,j)\exp(-\int_0^t q_i(Y_s)ds)dt.\end{aligned}$$

As a result, for $t\in[0,T]$,

$$\mathbb{P}_{\phi,i}\big\{\lambda_1\in dt,\beta(\lambda_1)=j\big|\mathcal{F}_T^W\big\}=q_{ij}(Y_t)\exp(-\int_0^t q_i(Y_s)ds)dt.$$

Thus,

$$\begin{aligned}&\mathbb{E}_{\phi,i}\Big[g(Y_{(1)},\lambda_1,\beta_1)\mathbf{1}_{\{\lambda_1\le T\}}\Big|\mathcal{F}_T^W\Big]\\ &=\sum_{j=1,j\neq i}^{\infty}\int_0^T g(Y_t,t,j)\mathbb{P}_{\phi,i}\big\{\lambda_1\in dt,\beta(\lambda_1)=j\big|\mathcal{F}_T^W\big\}\\ &=\sum_{j=1,j\neq i}^{\infty}\int_0^T g(Y_t,t,j)q_{ij}(Y_t)\exp(-\int_0^t q_i(Y_s)ds)dt\end{aligned}$$

as desired. □

*The proof of Proposition 2.10.* First, we prove (2.12) for the case $l=0$. Since $(X_t,\alpha(t))=(Y_t,\lambda(t))$ up to the moment $\alpha_1=\lambda_1$, we have

$$\begin{aligned}&\mathbb{E}_{\phi,i}\Big[f(X_T,\alpha(T))\mathbf{1}_{\{\tau_1>T\}}\Big]=\mathbb{E}_{\phi,i}\Big[f(Y_T,i)\mathbf{1}_{\{\lambda_1>T\}}\Big]\\ =&\mathbb{E}_{\phi,i}\Big[\mathbb{E}_{\phi,i}\big(f(Y_T,i)\mathbf{1}_{\{\lambda_1>T\}}|\mathcal{F}_T^W\big)\Big]=\mathbb{E}_{\phi,i}\Big[f(Y_T,i)\mathbb{E}_{\phi,i}\big(\mathbf{1}_{\{\lambda_1>T\}}|\mathcal{F}_T^W\big)\Big]\\ =&\mathbb{E}_{\phi,i}\Big[f(Y_T,i)\exp\Big(-\int_0^T q_i(Y_s)ds\Big)\Big],\end{aligned}\tag{A.4}$$

where the last equality is consequence of (i) of Lemma 2.9. Since $\gamma(\cdot)$ and $Y(\cdot)$ are independent, $Z_t = Y_t$ up to the moment $\theta_1$ and $\mathbb{P}_{\phi,i}\{\theta_1 > T\} = \exp(-T)$, we obtain

$$\begin{aligned}
&\mathbb{E}_{\phi,i}\Big(f(Z_T,\gamma(T))\mathbf{1}_{\{\theta_1>T\}}\exp\{-\int_0^T q_i(Z_s)ds\}\Big)\\
&=\mathbb{E}_{\phi,i}\Big(f(Y_T,i)\mathbf{1}_{\{\theta_1>T\}}\exp\{-\int_0^T q_i(Y_s)ds\}\Big)\\
&=\mathbb{P}_{\phi,i}\{\theta_1>T\}\mathbb{E}_{\phi,i}\Big(f(Y_T,i)\exp\{-\int_0^T q_i(Y_s)ds\}\Big)\\
&=\exp(-T)\mathbb{E}_{\phi,i}\Big(f(Y_T,i)\exp\{-\int_0^T q_i(Y_s)ds\}\Big).
\end{aligned} \tag{A.5}$$

From (A.4) and (A.5), we have for $t \in [0,T]$ that

$$\mathbb{E}f(X_T,\alpha(T))\mathbf{1}_{\{\tau_1>T\}} = \exp(T)\mathbb{E}\Big(f(Z_T,\gamma(T))\mathbf{1}_{\{\theta_1>T\}}\exp\{-\int_0^T q_i(Z_s)ds\}\Big). \tag{A.6}$$

We now prove (2.12) for $l = 1$. Let $g(\phi,t,i)$, $\tilde{g}(\phi,t,i) : \mathcal{C}\times[0,\infty)\times\mathbb{Z}_+ \to \mathbb{R}$ be bounded measurable functions and $g(\phi,t,i) = \tilde{g}(\phi,t,i) = 0$ if $t > T$. It follows from (ii) of Lemma 2.9 that

$$\begin{aligned}
&\mathbb{E}_{\phi,i}g(X_{(1)},\tau_1,\alpha_1) = \mathbb{E}_{\phi,i}g(Y_{(1)},\lambda_1,\beta_1)\\
&=\sum_{i_1\neq i}\int_0^T \mathbb{E}_{\phi,i}\Big(g(Y_t,t,i_1)q_{ii_1}(Y_t)\exp(-\int_0^t q_i(Y_s)ds)\Big)dt.
\end{aligned} \tag{A.7}$$

On the other hand,

$$\begin{aligned}
&\mathbb{E}_{\phi,i}\Big[\tilde{g}(Z_{(1)},\theta_1,\gamma_1)\exp(-\int_0^{\theta_1} q_i(Z_s)ds)\Big]\\
&=\mathbb{E}_{\phi,i}\Big[\tilde{g}(Y_{(1)},\theta_1,\gamma_t)\exp(-\int_0^{\theta_1} q_i(Y_s)ds)\Big]\\
&=\sum_{i_1\neq i}\int_0^T \mathbb{E}_{\phi,i}\Big(\tilde{g}(Y_t,t,i_1)\exp(-\int_0^t q_i(Y_s)ds)\Big)\mathbb{P}_{\phi,i}\{\theta_1\in dt,\gamma_1=i_1\}\\
&=\sum_{i_1\neq i}\int_0^T \mathbb{E}_{\phi,i}\Big(\tilde{g}(Y_t,t,i_1)\exp(-\int_0^t q_i(Y_s)ds)\Big)\rho_{ii_1}\exp(-t)dt.
\end{aligned} \tag{A.8}$$

Substituting $\tilde{g}(\phi,t,i) = g(\phi,t,i)\exp(t) \times \dfrac{q_{ii}(\phi)}{\rho_{ii}}$ into (A.8), we have

$$\begin{aligned}
&\mathbb{E}_{\phi,i}\Big[g(Z_{(1)},\theta_1,\gamma_1)\exp(\theta_1) \times \frac{q_{i\gamma_1}(Z_s)}{\rho_{i\gamma_1}}\exp(-\int_0^{\theta_1} q_i(Z_s)ds)\Big] \\
&\quad= \sum_{i_1\neq i}\int_0^T \mathbb{E}_{\phi,i}\Big[g(Y_t,t,i_1)\exp(t)\frac{q_{ii_1}(Y_t)}{\rho_{ii_1}}\exp(-\int_0^t q_i(Y_s)ds)\Big]\rho_{ii_1}\exp(-t)dt \qquad \text{(A.9)} \\
&\quad= \sum_{i_1\neq i}\int_0^T \mathbb{E}_{\phi,i}\Big[g(Y_t,t,i_1)q_{ii_1}(Y_t)\exp(-\int_0^t q_i(Y_s)ds)\Big]dt.
\end{aligned}$$

It follows from (A.7) and (A.9) that

$$\begin{aligned}
&\mathbb{P}_{\phi,i}\{\tau_1 \in dt, \alpha_1 = i_1, X_{(1)} \in d\phi_1\} \qquad \text{(A.10)} \\
&\quad= \mathbb{E}_{\phi,i}\Big[\mathbf{1}_{\{\theta_1\in dt,\gamma_1=i_1,Z_{(1)}\in d\phi_1\}}\exp(t) \times \frac{q_{ii_i}(Z_t)}{\rho_{ii_1}}\exp(-\int_0^t q_i(Z_s)ds)\Big].
\end{aligned}$$

We now use the strong Markov property of $(X_t,\alpha(t))$ and $(Z_t,\gamma(t))$, (A.10) as well as (A.6) with $\phi, i, T$ replaced by $\phi_1, i_1, T-t$, respectively;

$$\begin{aligned}
&\mathbb{E}_{\phi,i} f(X_T,\alpha(T))\mathbf{1}_{\{\tau_1\le T<\tau_2\}}\mathbf{1}_{\{\alpha_1=i_1\}} \\
&= \int_0^T\int_{\mathcal{C}}\Big[\mathbb{P}\{\tau_1\in dt,\alpha_1=i_1,X_{(1)}\in d\phi_1\} \times \mathbb{E}_{\phi_1,i_1} f(X_{T-t},\alpha(T-t))\mathbf{1}_{\{\tau_1>T-t\}}\Big] \\
&= \int_0^T\int_{\mathcal{C}}\Big[\mathbb{E}_{\phi,i}\Big(\mathbf{1}_{\{\theta_1\in dt,\gamma_1=i_1,Z_{(1)}\in d\phi_1\}}\exp(t) \times \frac{q_{ii_i}(Z_s)}{\rho_{ii_1}}\exp(-\int_0^t q_i(Z_s)ds)\Big) \qquad \text{(A.11)} \\
&\qquad \times \exp(T-t)\mathbb{E}_{\phi_1,i_1} f(Z_{T-t},\gamma(T-t))\mathbf{1}_{\{\theta_1>T-t\}}\exp\{-\int_0^{T-t} q_{i_1}(Z_s)ds\}\Big] \\
&= \exp(T)\mathbb{E}_{\phi,i}\Big[f(Z_T,\gamma(T))\mathbf{1}_{\{\theta_1\le T<\theta_2\}}\mathbf{1}_{\{\gamma_1=i_1\}}\exp\Big(-\int_{\theta_1}^T q_{i_1}(Z_s)ds\Big) \\
&\qquad \frac{q_{ii_1}(Z_{(1)})}{\rho_{ii_1}}\exp\Big(-\int_0^{\theta_1} q_i(Z_s)ds\Big)\Big].
\end{aligned}$$

We have already proved (2.12) for $l = 0,1$. Using the same argument, the induction, and the strong Markov property of $(X_t,\alpha(t))$ and $(Z_t,\gamma(t))$, we can obtain (2.12) for any $l \in \mathbb{Z}_+$.

□

*The proof of Lemma 2.12.* By (2.5), we can find $m \in \mathbb{Z}_+$ such that

$$\mathbb{P}_{\phi,i_0}\Big\{\tau_{m+1} < T\Big\} < \frac{\Delta}{2},\ \forall\,(\phi, i) \in \mathcal{C} \times \mathbb{Z}_+. \tag{A.12}$$

Now, let $\varepsilon = \varepsilon(\Delta) > 0$ (to be specified later). In view of [32, Theorem 4.3, p. 61], for each $i \in \mathbb{Z}_+$, there is a constant $C_i$ such that

$$\mathbb{E}_{\phi,i_0}|Y(t) - Y(s)|^6 \le C_i|t - s|^3\, \forall t, s \in [0, T],\ \forall\,\|\phi\| \le R + 1. \tag{A.13}$$

By the Kolmogorov-Centsov theorem (see [21, Theorem 2.8]), there is a positive random variable $h_i^\phi(\omega)$ such that

$$\mathbb{P}_{\phi,i_0}\Big\{\sup_{t,s\in[0,T],0<t-s<h_i^\phi(\omega)} \frac{|Y^{\phi,i}(t) - Y(s)|}{(t-s)^{0.25}} \le 4\Big\} = 1.$$

Since $C_i$ in (A.13) does not depend on $\phi \in \{\psi : \|\psi\| \le R+1\}$, it can be seen from the proof of the Kolmogorov-Centsov theorem that for any $\varepsilon > 0$, there is a constant $h_i > 0$ satisfying

$$\mathbb{P}_{\phi,i_0}\Big\{\sup_{t,s\in[0,T],0<t-s<h_i} \frac{|Y(t) - Y(s)|}{(s-t)^{0.25}} \le 4\Big\} > 1 - \varepsilon,\ \forall\,\|\phi\| \le R + 1. \tag{A.14}$$

Without loss of generality, we can choose $h_{i+1} < h_i, \forall i \in \mathbb{Z}_+$. Let

$$\mathcal{H}_{i,T} = \Big\{\psi(\cdot) \in \mathcal{C}([0, T], \mathbb{R}) : \|\psi\| \le R + 1 \text{ and } \sup_{t,s\in[0,T],0<t-s<h_i} \frac{|\psi(s) - \psi(t)|}{(s-t)^{0.25}} \le 4\Big\},$$

and

$$\mathcal{H}_i = \Big\{\psi(\cdot) \in \mathcal{C} : \|\psi\| \le R + 1 \text{ and } \sup_{t,s\in[-r,0],0<t-s<h_i} \frac{|\psi(s) - \psi(t)|}{(s-t)^{0.25}} \le 4\Big\}.$$

Hence $\mathcal{H}_{i+1,T} \supset \mathcal{H}_{i,T}$ and $\mathcal{H}_{i+1} \supset \mathcal{H}_i$. For $d > 0$ and a compact set $\mathcal{K} \subset \mathcal{C}$, we define

$$\mathcal{K}_d := \{\psi \in \mathcal{C} : \exists \phi \in \mathcal{K} \text{ such that } \|\psi - \phi\| < d\}.$$

Define $\mathcal{K}^0 = \{\psi(\cdot) = \phi_0(\cdot) + c : c \in \mathbb{R}^n, |c| \leq 1\}$, which is compact, and $\mathcal{K}^1 = \mathcal{K}^0 \uplus \mathcal{H}_{i_0}$. For each $\phi \in \mathcal{K}^1$, there is $n_{\phi,i_0} > i_0$ such that

$$\sum_{k=n_{\phi,i_0}+1}^{\infty} q_{i_0,k}(\phi) = q_{i_0}(\phi) - \sum_{k=1,k\neq i_0}^{n_{\phi,i_0}} q_{i_0 k}(\phi) < \frac{\varepsilon}{2}.$$

By the continuous of $q_{i_0}$ and $q_{i_0 k}(\phi)$, there is a $d_{\phi,i_0} > 0$ such that

$$\sum_{k=n(\phi)+1}^{\infty} q_{i_0,k}(\phi') = q_{i_0}(\phi') - \sum_{k=1,k\neq i_0}^{n_{\phi,i_0}} q_{i_0 k}(\phi') < \varepsilon \,\forall\, \|\phi' - \phi\| < d_{\phi,i_0}.$$

Since $\mathcal{K}^1$ is compact, there exist $n_1 > 0$ and $d_1 > 0$ such that

$$\sum_{k=n_1+1}^{\infty} q_{i_0,k}(\phi) < \varepsilon \,\forall \phi \in \mathcal{K}^1_{d_1}.$$

Define $\mathcal{K}^2 = \mathcal{K}^1 \uplus \mathcal{H}_{n_1}$. Using the compactness of $\mathcal{K}^2$, there exist $n_2 > n_1$ and $d_2 \in (0, d_1]$ such that

$$\sum_{k=n_2+1}^{\infty} q_{i,k}(\phi) < \varepsilon \,\forall i \in N_{n_1}, \phi \in \mathcal{K}^2_{d_2}.$$

Continuing this way, for $\mathcal{K}^m = \mathcal{K}^{m-1} \uplus \mathcal{H}_{n_{m-1}}$, there exists $n_m > n_{m-1}$ and $d_m \in (0, d_{m-1}]$ such that

$$\sum_{k=n_m+1}^{\infty} q_{i,k}(\phi) < \varepsilon \,\forall i \in N_{n_{m-1}}, \phi \in \mathcal{K}^m_{d_m}.$$

Set $\mathcal{K}^{\phi,1} = \{\phi\} \uplus \mathcal{H}_{i_0}$ and $\mathcal{K}^{\phi,k} = \mathcal{K}^{\phi,k-1} \uplus \mathcal{H}_{n_{k-1}}$ for $\phi \in \mathcal{C}$ and $k = 2, \ldots, m$. It is not difficult to verify that

$$\mathcal{K}^{\phi,k} \subset \mathcal{K}^k_{d_k} \,\forall\, k = 1, \ldots, m, \text{ for } \|\phi - \phi_0\| < \frac{d_m}{2}. \tag{A.15}$$

Denote by $\{Y(\cdot) \in \mathcal{H}_{n_0,T}\}$ the event $\{t \in [0,T] \mapsto Y(t)$ is a function belonging to $\mathcal{H}_{n_0,T}\}$.

Clearly, if $Y(\cdot) \in \mathcal{H}_{i_0,T}$, then $Y_t \in K^{\phi,1}\, \forall t \in [0,T]$. Thus, we can proceed as follows:

$$\begin{aligned}
&\mathbb{P}_{\phi,i_0}\{\tau_1 \le T, \big(X_{(1)}, \alpha_1\big) \notin \mathcal{K}^{\phi,1} \times N_{n_1}\} \\
&\quad =\mathbb{P}_{\phi,i_0}\big(\{\tau_1 \le T, \alpha_1 > n_1\} \cup \{\tau_1 \le T, X(\tau_1) \notin \mathcal{K}^{\phi,1}\}\big) \\
&\quad =\mathbb{P}_{\phi,i_0}\big(\{\lambda_1 \le T, \beta_1 > n_1\} \cup \{\lambda_1 \le T, Y(\lambda_1) \notin \mathcal{K}^{\phi,1}\}\big) \\
&\quad \le\mathbb{P}_{\phi,i_0}\{\lambda_1 \le T, Y(\cdot) \in \mathcal{H}_{n_0,T}, \beta_1 > n_1\} + \mathbb{P}\{Y(\cdot) \notin \mathcal{H}^T_{n_0}\} \\
&\quad \le\mathbb{E}_{\phi,i_0}\big[\mathbb{E}\big(\mathbf{1}_{\{Y(\cdot)\in\mathcal{H}_{n_0,T}\}}\mathbf{1}_{\{\lambda_1\le T,\beta_1>n_1\}}|\mathcal{F}^W_T\big)\big] + \varepsilon \\
&\quad =\mathbb{E}_{\phi,i_0}\big[\mathbf{1}_{\{Y(\cdot)\in\mathcal{H}_{n_0,T}\}}\mathbb{E}\big(\mathbf{1}_{\{\lambda_1\le T,\beta_1>n_1\}}|\mathcal{F}^W_T\big)\big] + \varepsilon \\
&\quad =\mathbb{E}_{\phi,i_0}\big[\mathbf{1}_{\{Y(\cdot)\in\mathcal{H}_{n_0,T}\}}\int_0^T \sum_{i>n_1} q_{i_0,i}(Y_t)\exp\big(-\int_0^t q_{i_0}(Y_s)ds\big)dt\big] + \varepsilon \\
&\quad \le\mathbb{E}_{\phi,i_0}\big[\mathbf{1}_{\{Y(\cdot)\in\mathcal{H}_{n_0,T}\}}\int_0^T \varepsilon dt\big] + \varepsilon \le (T+1)\varepsilon.
\end{aligned} \tag{A.16}$$

Similarly, if $(\phi_1, i_1) \in \mathcal{K}^{\phi,1} \times N_1$, then $\mathbb{P}_{\phi_1,i_1}\{\tau_1 \le T, \big(X_{(1)}, \alpha_1\big) \notin \mathcal{K}^{\phi,2} \times N_{n_2}\} \le (T+1)\varepsilon$.

Using the strong Markov property of $(X_t, \alpha(t))$, we obtain

$$\begin{aligned}
&\mathbb{P}_{\phi,i_0}\Big\{\tau_1 < T, \big(X_{(1)}, \alpha_1\big) \in \mathcal{K}^{\phi,1} \times N_{n_1}, \tau_2 \le T, \big(X_{(2)}, \alpha_2\big) \notin \mathcal{K}^{\phi,2} \times N_{n_2}\Big\} \\
&\quad \le\mathbb{P}_{\phi,i_0}\{\tau_1 < T, \big(X_{(1)}, \alpha_1\big) \in \mathcal{K}^{\phi,1} \times N_{n_1}\} \\
&\quad \times \mathbb{P}_{\phi,i_0}\Big[\tau_2 \le T + \tau_1, \big(X_{(2)}, \alpha_2\big) \notin \mathcal{K}^{\phi,2} \times N_{n_2}\Big|\tau_1 < T, \big(X_{(1)}, \alpha_1)\big) \in \mathcal{K}^{\phi,1} \times N_{n_1}\Big] \\
&\quad \le \sup_{(\phi_1,i_1)\in\mathcal{K}^\phi_1\times N_{n_1}} \mathbb{P}_{\phi_1,i_1}\{\tau_1 \le T, \big(X_{(1)}, \alpha_1\big) \notin \mathcal{K}^{\phi,2} \times N_{n_2}\} \le (T+1)\varepsilon.
\end{aligned}$$

Continuing this way, we can show for any $k = 1, \ldots, m$ that

$$\begin{aligned}
&\mathbb{P}_{\phi,i_0}\Big\{\tau_k \le T, \big(X_{\tau_k}, \alpha_k\big) \notin \mathcal{K}^\phi_k \times N_{n_k}, \big(X_{(j)}, \alpha_j\big) \in \mathcal{K}^\phi_j \times N_{n_j}, j = 1, \ldots, k-1\Big\} \\
&\quad \le (T+1)\varepsilon.
\end{aligned} \tag{A.17}$$

Consequently,

$$\mathbb{P}_{\phi,i_0}\Big\{\exists k=1,\ldots,m:\tau_k\le T \text{ and } \big(X_{(k)},\alpha_k\big)\notin\mathcal{K}_k^{\phi}\times N_{n_k}\Big\}\le (T+1)m\varepsilon.$$

Hence, if we choose $\varepsilon=\dfrac{1}{2m(T+1)}\Delta$,

$$\begin{aligned}&\mathbb{P}_{\phi,i_0}\Big\{\forall k=1,\ldots,m:\tau_k> T \text{ or } \alpha_k\in N_{n_k}\Big\}\\&\qquad\ge\mathbb{P}\Big\{\forall k=1,\ldots,m:\tau_k> T \text{ or } \big(X_{(k)},\alpha_k\big)\in\mathcal{K}_k^{\phi}\times N_{n_k}\Big\}\ge 1-\frac{\Delta}{2}.\end{aligned}\tag{A.18}$$

It follows from (A.12) and (A.18) that

$$\mathbb{P}_{\phi,i_0}\Big(\{\tau_{m+1}>T\}\cap\Big\{\forall k=1,\ldots,m:\tau_k>T \text{ or } \alpha_k\in N_{n_k}\Big\}\Big)\ge 1-\Delta.$$

It is easily verified that if $\omega\in\{\tau_{m+1}>T\}\cap\{\forall k=1,\ldots,m:\tau_k>T \text{ or } \alpha_k\in N_{n_k}\}$, then $\alpha(t)\in N_{n_m},\forall t\in[0,T]$. The assertion of the lemma is proved. □

# APPENDIX B: SUPPLEMENTS FOR CHAPTER 3

Let $Y^{x,i}(t)$ be the solution to

$$dY(t) = b(Y(t), i)dt + \sigma(Y(t), i)dW(t),\ \ t \geq 0 \tag{B.1}$$

with initial condition $(x, i) \in \mathbb{R}^n \times \mathbb{Z}_+$. For $(\phi, i) \in \mathcal{C} \times \mathbb{Z}_+$, we denote by $Y^{\phi,i}(t), t \geq -r$ be the process satisfying $Y^{\phi,i}(t) = \phi$ if $t \in [-r, 0]$ and $Y^{\phi,i}(t)$ solves (B.1) for $t > 0$. Clearly $Y^{\phi,i}(t) = Y^{\phi(0),i}(t)$ for $t \geq 0$. Let $\beta^{\phi,i}$ be the solution to

$$\beta^{\phi,i}(t) = i + \int_0^t \int_{\mathbb{R}} h(Y_t^{\phi,i}, \beta^{\phi,i}(t-), z)\mathfrak{p}(dt, dz), t \geq 0 \tag{B.2}$$

satisfying $Y^{\phi,i}(t) = \phi(t)$ in $[-r, 0]$ and $\beta^{\phi,i}(0) = i$. Let $\xi_1^{\phi,i}(t)$ and $\lambda_1^{\phi,i}(t)$ be the first jump times of $\alpha^{\phi,i}(t)$ and $\beta^{\phi,i}(t)$, respectively. Clearly we have that

$$X^{\phi,i}(t) = Y^{\phi,i}(t),\ \alpha^{\phi,i}(t) = \beta^{\phi,i}(t)\ \text{ up to }\ \xi_1^{\phi,i}(t) = \lambda_1^{\phi,i}(t). \tag{B.3}$$

*Proof of Lemma* 3.2. Since $q_{ij}(\cdot)$ is continuous, there is an $\varepsilon \in (0, 1)$ such that $q_{ij}(\psi) > 0$ given that $\|\psi - \phi\| < \varepsilon$. Let $M_\phi = \sup_{\psi\in\mathcal{C},\|\psi-\phi\|<1}\{q_i(\psi)\} < \infty$. Let $\delta_1 > 0$ such that

$$|\phi(s) - \phi(s')| < \frac{\varepsilon}{5}\ \text{ provided }\ |s - s'| < \delta_1, s, s' \in [-r, 0]. \tag{B.4}$$

Under either Assumption 2.3 or Assumption 2.4, standard arguments show that there exists a sufficiently small $\delta_2 \in (0, \delta_1]$ satisfying

$$\mathbb{P}_{\psi,i}\left\{|Y(t) - \psi(0)| \leq \frac{\varepsilon}{5}\,\forall t \in [0, \delta_2]\right\} \geq \frac{1}{2}, \forall\psi \in \mathcal{C}, \|\psi - \phi\| < \varepsilon, \tag{B.5}$$

and

$$\mathbb{P}_{\psi',j}\left\{|Y(t) - \psi'(0)| \leq \frac{\varepsilon}{5}\,\forall t \in [0, \delta_2]\right\} \geq \frac{1}{2}, \forall\psi' \in \mathcal{C}, \|\psi' - \phi\| < \varepsilon. \tag{B.6}$$

In view of (B.4), it can be checked that

$$\|Y_t^{\psi,i} - \phi\| \le \frac{3\varepsilon}{5}\, \forall t \in [0, \delta_2] \text{ if } |Y^{\psi,i}(t) - \psi(0)| \le \frac{\varepsilon}{5}\, \forall t \in [0, \delta_2] \text{ and } \|\psi - \phi\| < \frac{\varepsilon}{5} \tag{B.7}$$

and

$$\|Y_t^{\psi',j} - \phi\| \le \varepsilon\, \forall t \in [0, \delta_2] \text{ if } |Y^{\psi',j}(t) - \psi'(0)| \le \frac{\varepsilon}{5}\, \forall t \in [0, \delta_2] \text{ and } \|\psi' - \phi\| < \frac{3\varepsilon}{5}. \tag{B.8}$$

By virtue of (B.5), (B.7), and Lemma 2.9, for $\psi \in \mathcal{C}$, $\|\psi - \phi\| < \dfrac{\varepsilon}{5}$ we have

$$\begin{aligned}
&\mathbb{P}_{\psi,i}\left\{\|Y_{\lambda_1} - \phi\| \le \frac{3\varepsilon}{5} \text{ and } \lambda_1 < \delta_2, \beta(\lambda_1) = j\right\} \\
&\quad = \mathbb{E}_{\psi,i} \int_0^{\delta_2} \mathbf{1}_{\{\|Y_t - \phi\| \le \frac{3\varepsilon}{5}\}} q_{i,j}(Y_t) \exp\left(-\int_0^t q_i(Y_s)ds\right) dt \\
&\quad \ge \mathbb{E}_{\psi,i}\left[\mathbf{1}_{\{|Y(u) - \psi(0)| \le \frac{\varepsilon}{5}\ \forall u \in [0,\delta_2]\}} \int_0^{\delta_2} q_{i,j}(Y_t) \exp\left(-\int_0^t q_i(Y_s)ds\right) dt\right] \\
&\quad \ge \frac{\delta_2}{2} \inf_{\phi' \in \mathcal{C}: \|\phi' - \phi\| \le \frac{3\varepsilon}{5}} \{q_{i,j}(\phi')\} \times \inf_{\phi' \in \mathcal{C}: \|\phi' - \phi\| \le \frac{3\varepsilon}{5}} \left\{\exp\left(-\int_0^{\delta_2} q_i(\phi'(s))ds\right) dt\right\} := p_1 > 0.
\end{aligned} \tag{B.9}$$

Now, we have from the Markov property that

$$\begin{aligned}
&\mathbb{P}_{\psi,i}\{\|X_{\delta_2} - \phi\| < \varepsilon, \alpha(\delta_2) = j\} \\
&\quad \ge \mathbb{P}_{\psi,i}\left\{\xi_1 < \delta_2, \alpha_1 = j, \|X_{\xi_1} - \phi\| < \frac{3\varepsilon}{5}\right\} \\
&\qquad \times \mathbb{P}_{\psi,i}\Big\{\|X_{\delta_2} - \phi\| < \varepsilon, \xi_2 > \delta_2 \Big| \xi_1 < \delta_2, \alpha_1 = j, \|X_{\xi_1} - \psi\| < \varepsilon\Big\}.
\end{aligned} \tag{B.10}$$

By (B.5) and (B.8), if $\|\psi' - \phi\| \le \frac{3\varepsilon}{5}$, then

$$
\begin{aligned}
&\mathbb{P}_{\psi',j}\{\|X_t - \phi\| < \varepsilon \, \forall\, t \in [0,\delta_2], \xi_1 > \delta_2\} \\
&\quad = \mathbb{P}_{\psi',j}\{\|Y_t - \phi\| < \varepsilon \, \forall\, t \in [0,\delta_2], \lambda_1 > \delta_2\} \\
&\quad \ge \mathbb{E}_{\psi',j}\left[1_{\{\|Y_t-\phi\|<\varepsilon\,\forall t\in[0,\delta_2]\}} \exp\left(-\int_0^{\delta_2} q_j(Y_s)ds\right)\right] \\
&\quad \ge \mathbb{P}_{\psi',j}\left\{|Y(t) - \psi'(0)| \le \frac{\varepsilon}{5} \, \forall t \in [0,\delta_2]\right\} \times \inf_{\phi'\in\mathcal{C}:\|\phi'-\phi\|\le\varepsilon}\left\{\exp\left(-\int_0^{\delta_2} q_i(\phi'(s))ds\right)\right\} \\
&\quad := p_2 > 0.
\end{aligned}
\tag{B.11}
$$

By the strong Markov property of $(X_t, \alpha(t))$, applying estimates (B.9) and (B.11) to (B.10), we obtain

$$
\sup_{\psi\in\mathcal{C}:\|\psi-\phi\|\le\frac{\varepsilon}{5}} \mathbb{P}_{\psi,i}\{\|X_{\delta_2} - \phi\| < \varepsilon, \alpha(\delta_2) = j\} > p_1 p_2 > 0.
$$

□

*Proof of Lemma* 3.3. Using the Kolmogorov-Centsov theorem [21, Theorem 2.8], for each $i \in \mathbb{Z}_+$ and $R > 0$, there exists an $h_{i,R} > 0$ such that

$$
\mathbb{P}_{\phi,i}\left\{\sup_{t,s\in[0,r],0<t-s<h_{i,R}} \frac{|Y(t) - Y(s)|}{(s-t)^{0.25}} \le 4\right\} > \frac{1}{2}, \ \forall\, \|\phi\| \le R. \tag{B.12}
$$

Let

$$
\mathcal{A} = \left\{\phi \in \mathcal{C} : |\phi(-r)| \le R, \sup_{t,s\in[-r,0],0<t-s<h_{i,R}} \frac{|\phi(t) - \phi(s)|}{(s-t)^{0.25}} \le 4\right\}.
$$

Let $R' > R$ such that $\|\phi\| < R'$ for any $\phi \in \mathcal{A}$ and $M_{R'} = \sup_{\|\phi\|<R'}\{q_i(\phi)\} < \infty$. For

$\|\phi\| \leq R$, we have that

$$\begin{aligned}\mathbb{P}_{\psi,i}\{X_r \in \mathcal{A}, \xi_1 > r\} =& \mathbb{P}_{\psi,i}\{Y_r \in \mathcal{A}, \lambda_1 > r\} \\ =& \mathbb{E}_{\psi,i}\left[\mathbf{1}_{\{Y_r\in\mathcal{A}\}}\int_0^r \exp(-q_i(Y_t))dt\right] \\ \geq& \mathbb{E}_{\psi,i}\mathbf{1}_{\{Y_r\in\mathcal{A}\}}\exp(-rM_{R'}) \\ \geq& 0.5\exp(-rM_{R'}),\end{aligned}$$

which implies (3.6).

To prove (3.7), note that $\sup_{(x,i)\in\mathbb{R}^n\times\mathbb{Z}_+}\mathbb{E}_{x,i}\tau_k < \infty$ where $\tau_k = \inf\{t \geq 0 : |Y(t)| \geq k\}$, (see e.g., [24, Corrolary 3.3] or [57, Theorem 3.1]). Thus, there is a $T > 0$ such that

$$\mathbb{P}_{x,i}\Big\{\tau_k < T\Big\} > \frac{1}{2},\ \forall x \in \mathbb{R}^n.$$

Denote $\widetilde{\tau}_k = \inf\{t \geq 0 : \|X_t\| \geq k\}$. For $\phi \in \mathcal{C}$ with $\|\phi\| \leq R < k$ we have from (B.3) and Lemma 2.9 that

$$\begin{aligned}\mathbb{P}_{\phi,i}\Big\{\widetilde{\tau}_k < T\Big\} \geq& \mathbb{P}_{\phi,i}\Big\{\widetilde{\tau}_k < T, \alpha(t) = i \text{ for } t \in [0,\widetilde{\tau}_k)\Big\} \\ =& \mathbb{P}_{\phi,i}\Big\{\tau_k < T, \beta(t) = i \text{ for } t \in [0,\tau_k)\Big\} \\ =& \mathbb{E}_{\phi,i}\left[\mathbf{1}_{\{\tau_k<T\}}\exp\left(-\int_0^{\tau_k} q_i(Y_s)ds\right)\right] \\ \geq& \exp\left(-M_kT\right)\mathbb{E}_{\phi,i}\mathbf{1}_{\{\tau_k<T\}} \geq 0.5\exp\left(-M_kT\right),\end{aligned}$$

where $M_k = \sup_{\|\phi\|<k}\{q_i(\phi)\} < \infty$. The proof is therefore complete. □

We need an auxiliary lemma to obtain Lemma 3.4.

**Lemma B.1.** *Fix $i \in \mathbb{Z}_+$ and suppose $A(x,i)$ is elliptic uniformly in each compact subset of $\mathbb{R}^n$. For $D$ be a bounded open set in $\mathbb{R}^n$ and $K_1, K_2$ be open sets whose closures are contained* *in $D$. Then*

$$\inf_{\{\phi\in\mathcal{C}:\phi(0)\in K_1\}} \mathbb{P}_{\phi,i}\big(\{Y(T)\in K_2\}\cap\{Y(t)\in D\forall t\in[0,T]\}\big)>0 \tag{B.13}$$

*and there is a measure $\boldsymbol{\nu}$ on $\mathfrak{B}(\mathcal{C})$ such that*

$$\mathbb{P}_{\phi,i}\{Y_{T+r}\in\mathcal{B},Y(t)\in D,t\in[0,T+r]\}\geq\boldsymbol{\nu}(\mathcal{B}).$$

*Moreover, if $\mathcal{B}\subset\{\phi\in\mathcal{C}:\phi(t)\in D,t\in[-r,0]\}$ is an open set of $\mathcal{C}$, then $\boldsymbol{\nu}(\mathcal{B})>0$.*

*Proof.* For a bounded continuous function $f(x):\mathbb{R}^n\mapsto\mathbb{R}$ vanishing outside $K_2$, let $u_f(t,x)$ be the solution to

$$\begin{cases}\dfrac{\partial u}{\partial t}+\mathcal{L}_i u=0 \text{ in } D\times[0,T)\\ u\,(T,x)=f(x) \text{ on } D,\\ u(t,x)=0 \text{ on } \partial D\times[0,T]\,.\end{cases} \tag{B.14}$$

It is well known (see, e.g., [32, Theorem 2.8.2]) that

$$u_f(t,x)=\mathbb{E}_{x,i}\left[f\big(Y(T-t)\big)\mathbf{1}_{\{Y(s)\in D\,\forall s\in[0,T-t]\}}\right].$$

Let $g$ be a continuous function in $D$ such that $0\leq g(x)\leq 1\,\forall x\in D$, $g(x)=0$ outside $K_2$ and $g(x)>0$ for some $x\in K_2$. By the strong maximum principle for parabolic equations (see [13, Theorem 7.12]), $u_g(0,x)>0$ for all $x\in D$, which implies that

$$u_C:=\inf\{u_g(0,x):x\in K_1\}>0. \tag{B.15}$$

By the definition of $g(\cdot)$, we can obtain that

$$\mathbb{P}_{x,i}\left\{Y(T)\in K_2,Y(s)\in D\,\forall s\in[0,T]\right\}\geq u_g(0,x)\,\forall x\in D. \tag{B.16}$$

The first desired result follows from (B.15) and (B.16). Moreover, in view of Harnack's inequality (see [13, Theorem 7.10]), there is $\tilde{\rho}_i > 0$ such that $u_f(y,T) \geq \tilde{\rho}_i u_f(x_0, \frac{T}{2})$ for all $y \in K_2$ and $f$ being bounded continuous. Thus

$$\mathbb{E}_{x,i}\left[f\big(Y(T)\big)\mathbf{1}_{\{Y(s)\in D\,\forall s\in[0,T-t]\}}\right] \geq \rho_i \mathbb{E}_{x_0,i}\left[f\left(Y\left(0.5T\right)\right)\mathbf{1}_{\{Y(s)\in D\,\forall s\in[0,T-t]\}}\right]$$

for any bounded and continuous function $f$. Thus, we obtain that

$$\begin{aligned}
&\mathbb{P}_{x,i}\left\{Y(T)\in B \ \text{ and } \ Y(s)\in D\,\forall s\in[0,T]\right\}\\
&\geq \tilde{\rho}_i \mathbb{P}_{x_0,i}\left\{Y\left(0.5T\right)\in B, \ \text{ and } \ Y(s)\in D\,\forall s\in[0,0.5]\right\}\\
&\geq \rho_i \tilde{\nu}(B)
\end{aligned}$$

for any Borel set $B$, where

$$\nu(\cdot) = \mathbb{P}_{x_0,i}\left\{Y(0.5T)\in\cdot, \ \text{ and } \ Y(s)\in D\,\forall s\in[0,0.5T]\right\}$$

and $\rho_i = \tilde{\rho}_i \mathbb{P}_{x_0,i}\left\{Y(s)\in D\,\forall s\in[0,0.5T]\right\}$, which is positive due to (B.13). Denote $\widehat{\mathcal{D}} = \{\phi \in \mathcal{C} : \phi(t)\in D, t\in[-r,0]\}$. For any Borel set $\mathcal{B}\subset\mathcal{C}$, we have from the Markov property of $Y^i(t)$ that

$$\begin{aligned}
&\mathbb{P}_{x_0,i}\left\{Y_{T+r}\in\mathcal{B} \ \text{ and } Y(s)\in D\,\forall s\in[0,T+r]\right\}\\
&=\mathbb{P}\left\{Y_{T+r}\in\mathcal{B}\cap\widehat{\mathcal{D}}\middle|Y(T)=y\right\}\mathbb{P}_{\phi_0,i}\left\{Y(T)\in dy \ \text{ and } Y(s)\in D\,\forall s\in[0,T+r]\right\}\\
&\geq\rho_i\int_{y\in D}\mathbb{P}\left\{Y_{T+r}\in\mathcal{B}\cap\widehat{\mathcal{D}}\middle|Y(T)=y\right\}\tilde{\nu}(dy)\\
&=\boldsymbol{\nu}(\mathcal{B}\cap\widehat{\mathcal{D}}).
\end{aligned}$$

Now, let $\mathcal{B}$ be an open subset of $\widehat{\mathcal{D}}$. Denote $B = \{\phi(-r) : \phi\in\mathcal{B}\}$. Then $B$ is an open subset of $D$. By the support theorem (see [50, Theom 3.1])),

$$\mathbb{P}_{y,i}\left\{Y_{T+r}\in\mathcal{B}\right\}>0 \text{ for any } y\in B. \tag{B.17}$$

In light of (B.13),

$$\tilde{\nu}(B)=\mathbb{P}_{\phi_0,i}\left\{Y(0.5T)\in B, \text{ and } Y(s)\in D\,\forall s\in[0,0.5T]\right\}>0. \tag{B.18}$$

In view of (B.17) and (B.18),

$$\boldsymbol{\nu}(\mathcal{B})\geq\int_{y\in B}\mathbb{P}_{y,i}\left\{Y_{T+r}\in\mathcal{B}\right\}\tilde{\nu}(dy)>0$$

if $\mathcal{B}$ is an open subset of $\widehat{\mathcal{D}}$. □

*Proof of Lemma* 3.4. Let $D=\{x\in\mathbb{R}^n:|x|<R+1\}$, $K_1=K_2=\{x\in\mathbb{R}^n:|x|<R\}$ and $M_{R,i}=\sup\{q(\phi,i):\|\phi\|\leq R+1\}<\infty$. In view of Lemma B.1,

$$\mathbb{P}_{\phi,i^*}\left\{Y_{T+r}\in\mathcal{B},Y(t)<R+1,t\in[0,T+r]\right\}\geq\boldsymbol{\nu}(\mathcal{B})$$

where $\boldsymbol{\nu}(\cdot)$ is defined as in Lemma B.1 with $i$ replaced by $i^*$. Thus,

$$\begin{aligned}&\mathbb{P}_{\phi,i^*}\{X_{T+r}\in\mathcal{B}\text{ and }\alpha(T+r)=i^*\}\\&\quad\geq\mathbb{P}_{\phi,i^*}\{X_{T+r}\in\mathcal{B}\text{ and }\alpha(t)=i^*,\|X_t\|<R+1\,\forall\,t\in[0,T+r]\}\\&\quad\geq\mathbb{E}_{\phi,i^*}\left[\mathbf{1}_{\{Y_{T+r}\in\mathcal{B},\|Y_t\|<R\,\forall\,t\in[0,T+r]\}}\exp\left(-\int_0^{T+r}q_{i^*}(Y_t)dt\right)\right]\\&\quad\geq\exp(-M_{R,i}(T+r))\boldsymbol{\nu}(\mathcal{B}).\end{aligned}$$

The proof is complete. □

# APPENDIX C: SUPPLEMENTS FOR CHAPTER

*Proof of Lemma 4.5.* Since $V(0) = 0$ and $V(x)$ is continuous on $D$, we can find $h_* > 0$ such that $B_{h_*} \subset D$ and $V(x) \leq 1$ for any $x \in B_{h_*}$. Because $\tau_{h_1} \leq \tau_{h_2}$ if $h_1 \leq h_2$, it suffices to prove the Lemma for any $h \leq h_*$.

Since $g$ is continuously differentiable and $g(0) = 0$, there is $K_g > 0$ such that $g(z) \leq K_g|z|$ for $|z| \leq 1$. Thus, we have

$$\mathcal{L}_i V(x) \leq K_g \sup_{i\in\mathbb{Z}_+}\{|c_i|\}V(x),\ (x,i) \in B_{h_*} \times \mathbb{Z}_+$$

Letting $\tilde{K} = K_g \sup_{i\in\mathbb{Z}_+}\{|c_i|\}$, by Itô's formula,

$$\begin{aligned}\mathbb{E}_{x,i}V(X(t\wedge\tau_h)) \leq& V(x) + \tilde{K}\mathbb{E}_{x,i}\int_0^{t\wedge\tau_h} V(X(s))ds\\ \leq& V(x) + \tilde{K}\mathbb{E}_{x,i}\int_0^{t} \mathbb{E}_{x,i}V(X(s\wedge\tau_h))ds.\end{aligned}$$

By the Grownwall inequality, we can easily obtain

$$\mathbb{E}_{x,i}V(X(T\wedge\tau_h)) \leq V(x)e^{KT}.$$

Let $v_h = \inf\{V(x) : |x| = h\} > 0$. An application of Markov's inequality yields that

$$\mathbb{P}_{x,i}\{\tau_h \leq T\} \leq \frac{V(x)e^{KT}}{v_h}.$$

Since $V(0) = 0$ and $V$ is continuous on $D$, there is a $\tilde{h} > 0$ such that

$$\mathbb{P}_{x,i}\{\tau_h \leq T\} \leq \frac{V(x)e^{KT}}{v_h} \leq \varepsilon$$

for any $x \in B_{\tilde{h}}$ as desired. □

*Proof of Lemma 4.6.* It is easy to show that there exists some $K_2 > 0$ such that

$$|y|^k \exp(\theta y) \leq K_2(\exp(\theta_0 y) + \exp(-\theta_0 y)), k = 1, 2.$$

for $\theta \in \left[0, \frac{\theta_0}{2}\right]$, $y \in \mathbb{R}$. For any $y \in \mathbb{R}$, let $\xi(y)$ be a number lying between $y$ and 0 such that $\exp(\xi(y)) = \dfrac{e^y - 1}{y}$. Pick $\theta \in \left[0, \frac{\theta_0}{2}\right]$ and let $h \in \mathbb{R}$ such that $0 \leq \theta + h \leq \frac{\theta_0}{2}$. Then

$$\lim_{h \to 0} \frac{\exp((\theta + h)Y) - \exp(\theta Y)}{h} = Y \exp(\theta Y) \text{ a.s.,}$$

where $Y$ is as defined in Lemma 4.6, and

$$\left| \frac{\exp((\theta + h)Y) - \exp(\theta Y)}{h} \right| = |Y| \exp(\theta Y + \xi(hY)) \leq 2K_3[\exp(\theta_0 Y) + \exp(-\theta_0 Y)].$$

By the Lebesgue dominated convergence theorem,

$$\frac{d\mathbb{E}\exp(\theta Y)}{d\theta} = \lim_{h \to 0} \mathbb{E} \frac{\exp((\theta + h)Y) - \exp(\theta Y)}{h} = \mathbb{E} Y \exp(\theta Y).$$

Similarly,

$$\frac{d^2 \mathbb{E}\exp(\theta Y)}{d\theta^2} = \mathbb{E} Y^2 \exp(\theta Y).$$

As a result, we obtain

$$\frac{d\phi}{d\theta} = \frac{\mathbb{E} Y \exp(\theta Y)}{\mathbb{E}\exp(\theta Y)}$$

which implies

$$\frac{d\phi}{d\theta}(0) = \mathbb{E} Y$$

and

$$\frac{d^2\phi}{d\theta^2} = \frac{\mathbb{E} Y^2 \exp(\theta Y) \mathbb{E}\exp(\theta Y) - [\mathbb{E} Y \exp(\theta Y)]^2}{[\mathbb{E}\exp(\theta Y)]^2}.$$

By Hölder's inequality we have $\mathbb{E}Y^2\exp(\theta Y)\mathbb{E}\exp(\theta Y) \geq [\mathbb{E}Y\exp(\theta Y)]^2$ and therefore

$$\frac{d^2\phi}{d\theta^2} \geq 0\,, \forall\,\theta \in \left[0, \frac{\theta_0}{2}\right].$$

Moreover,

$$\begin{aligned}\frac{d^2\phi}{d\theta^2} \leq& \frac{\mathbb{E}Y^2\exp(\theta Y)}{\mathbb{E}\exp(\theta Y)}\\ \leq& \frac{K_3(\mathbb{E}\exp(\theta_0 Y) + \mathbb{E}\exp(-\theta_0 Y))}{\exp(\theta\mathbb{E}Y)}\\ \leq& \frac{K_3(\mathbb{E}\exp(\theta_0 Y) + \mathbb{E}\exp(-\theta_0 Y))}{\exp(-\theta_0|\mathbb{E}Y|)} := K_2,\end{aligned}$$

which concludes the proof. □

*Proof of Lemma 4.7.* Let $\overline{\tau}_n$ be the $n-$th jump moment of $\alpha(t)$. Let $T > 0$, In view of [32, Lemma 4.3.2], we have

$$\mathbb{P}_{x,i}\{X(t) = 0 \text{ for some } t \in [0, T \wedge \overline{\tau}_1]\} = 0 \text{ for any } x \neq 0, i \in \mathbb{Z}_+.$$

Since $X(T \wedge \overline{\tau}_1) \neq 0$ a.s., applying [32, Lemma 4.3.2] again yields

$$\mathbb{P}_{x,i}\{X(t) = 0 \text{ for some } t \in [T \wedge \overline{\tau}_1, T \wedge \overline{\tau}_2]\} = 0 \text{ for any } x \neq 0, i \in \mathbb{Z}_+.$$

Continuing this way, we have

$$\mathbb{P}_{x,i}\{X(t) = 0 \text{ for some } t \in [0, T \wedge \overline{\tau}_n]\} = 0 \text{ for any } x \neq 0, i \in \mathbb{Z}_+, n \in \mathbb{Z}_+. \tag{C.1}$$

In [34, Theorems 3.1 & 3.3], we have that $\lim_{n\to\infty} \overline{\tau}_n = \infty$. This and (C.1) imply

$$\mathbb{P}_{x,i}\{X(t) = 0 \text{ for some } t \in [0, T]\} = 0 \text{ for any } x \neq 0, i \in \mathbb{Z}_+, n \in \mathbb{Z}_+.$$

Since $T$ is taken arbitrarily, we obtain the desired result. □

**Lemma C.1.** *If the Markov chain $\widehat{\alpha}(t)$ is strongly exponentially ergodic with generator $\widehat{Q}$ and invariant probability measure $\boldsymbol{\nu} = (\nu_1, \nu_2, \dots)^\top$, then if $\mathbf{b} = (b_1, b_2, \dots)^\top$ is bounded satisfying $\sum \nu_i b_i = 0$, then, there exists a bounded vector $\mathbf{c} = (c_1, c_2, \dots)^\top$ such that $b_i = \sum \widehat{q}_{ji} c_j$.*

*Proof.* Let $\widehat{P}(t) = \widehat{p}_{ij}(t)$, where $\widehat{p}_{ij}(t) = \mathbb{P}\{\widehat{\alpha}(t) = j | \alpha(0) = i\}$, the transition matrix of $\widehat{\alpha}(t)$. Let $\mathbf{c} = (c_1, c_2, \dots)^\top$ where $c_i = \int_0^\infty [\nu_j b_j - \widehat{P}_{ij}(t) b_i] dt$. In view of (4.7), it is easy to see that $\mathbf{c}$ is bounded. Let $\mathbb{1} = (1, 1, \dots)$. We have

$$\begin{aligned}
\widehat{Q}\mathbf{c} &= \int_0^\infty \big[\widehat{Q}\boldsymbol{\nu}\mathbb{1}\mathbf{b} - \widehat{Q}\widehat{P}(t)\mathbf{b}\big] dt \\
&= -\int_0^\infty \widehat{Q}\widehat{P}(t)\mathbf{b} dt \\
&= -\int_0^\infty \widehat{P}(t)\mathbf{b} dt = -\widehat{P}(t)\mathbf{b}\Big|_0^\infty \\
&= -\mathbb{1}\boldsymbol{\nu}\mathbf{b} + \mathbf{b} = \mathbf{b}.
\end{aligned}$$

□

**Lemma C.2.** *Suppose that Assumption 4.3 and (4.8) hold. Then for any $T > 0$ and a bounded function $f : \mathbb{Z}_+ \mapsto \mathbb{R}$, we have*

$$\lim_{x \to 0} \sup_{i \in \mathbb{Z}_+, t \in [0,T]} \{|\mathbb{E}_{x,i} f(\alpha(t)) - \mathbb{E}_i f(\widehat{\alpha}(t)|\} = 0. \tag{C.2}$$

*Proof.* By the basic coupling method, we can consider the joint process $(X(t), \alpha(t), \widehat{\alpha}(t))$ as a switching diffusion where the diffusion $X(t) \in \mathbb{R}^n$ satisfies satisfying

$$dX(t) = b(X(t), \alpha(t))dt + \sigma(X(t), \alpha(t))dw(t) \tag{C.3}$$

and the switching part $(\alpha(t), \widehat{\alpha}(t)) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ has the generator $\widetilde{Q}(X(t))$ which is defined by

$$\widetilde{Q}(x)\widetilde{f}(k,l) = \sum_{j,i\in\mathbb{Z}_+} \widetilde{q}_{(k,l)(j,i)}(x)\left(\widetilde{f}(j,i) - \widetilde{f}(k,l)\right)$$
$$= \sum_{j\in\mathbb{Z}_+} [q_{kj}(x) - q_{lj}(0)]^+(\widetilde{f}(j,l) - \widetilde{f}(k,l))$$
$$+ \sum_{j\in\mathbb{Z}_+} [q_{lj}(0) - q_{kj}(x)]^+(\widetilde{f}(k,j) - \widetilde{f}(k,l))$$
$$+ \sum_{j\in\mathbb{Z}_+} [q_{kj}(x) \wedge q_{lj}(0)](\widetilde{f}(j,j) - \widetilde{f}(k,l)). \tag{C.4}$$

In what follows, we use the notation $\mathbb{E}_{x,i,j}$ and $\mathbb{P}_{x,i,j}$ to denote the corresponding conditional expectation and probability for the coupled process $(X(t), \alpha(t), \widehat{\alpha}(t))$ conditioned on $(X(0), \alpha(0), \widehat{\alpha}(0)) = (x, i, j)$. Let $\vartheta = \inf\{t \geq 0 : \alpha(t) \neq \widehat{\alpha}(t)\}$. Define $\tilde{g} : \mathbb{Z} \times \mathbb{Z} \mapsto \mathbb{R}$ by $\widetilde{g}(k,l) = \mathbf{1}_{\{k=l\}}$. By the definition of the function $\widetilde{g}$, we have

$$\widetilde{Q}(x)\widetilde{g}(k,k) = \sum_{j\in\mathbb{Z}_+, j\neq k} [q_{kj}(x) - q_{kj}(0)]^+ + \sum_{j\in\mathbb{Z}_+, j\neq k} [q_{kj}(0) - q_{kj}(x)]^+$$
$$= \sum_{j\in\mathbb{Z}_+, j\neq k} |q_{kj}(x) - q_{kj}(0)| =: \Xi(x,k). \tag{C.5}$$

For any $\varepsilon > 0$, let $h > 0$ such that $B_h \in D$ and $\sup_{(x,k)\in B_h\times\mathbb{Z}_+} \Xi(x,k) < \frac{\varepsilon}{2T}$. Applying Itô's formula and noting that $\alpha(t) = \widehat{\alpha}(t), t \leq \vartheta$,

we obtain that

$$\mathbb{P}_{x,i,i}\{\vartheta \leq T \wedge \tau_h\} = \mathbb{E}_{x,i,i}\widetilde{g}\left(\alpha(\vartheta \wedge T \wedge \tau_h), \widehat{\alpha}(\vartheta \wedge T \wedge \tau_h)\right)$$
$$= \mathbb{E}_{x,i,i}\int_0^{\vartheta\wedge T\wedge\tau_h} \widetilde{Q}(X(t))\widetilde{g}(\alpha(t), \widehat{\alpha}(t))dt$$
$$= \mathbb{E}_{x,i,i}\int_0^{\vartheta\wedge T\wedge\tau_h} \Xi(X(t), \alpha(t))dt$$
$$\leq T \sup_{(x,i)\in B_h\times\mathbb{Z}_+} \Xi(x,k) \leq \frac{\varepsilon}{2}. \tag{C.6}$$

Thus In view of Lemma 4.5, there is $\delta > 0$ such that $\mathbb{P}_{x,i,i}\{\tau_h \leq T\} \leq \frac{\varepsilon}{2}$. This and (C.6) derive

$$\mathbb{P}_{x,i,i}\{\vartheta \wedge \tau_h \leq T\} \leq \mathbb{P}_{x,i,i}\{\vartheta \leq T \wedge \tau_h\} + \mathbb{P}_{x,i,i}\{\tau_h \leq T\} \leq \varepsilon.$$

We have that

$$\begin{aligned} |\mathbb{E}_{x,i} f(\alpha(t)) - \mathbb{E}_{0,i} f(\alpha(t))| &= |\mathbb{E}_{x,i,i}\left[f(\alpha(t)) - f(\widehat{\alpha}(t))\right]| \\ &= \left|\mathbb{E}_{x,i,i} \mathbf{1}_{\{\vartheta \wedge \tau_h \leq t\}}\left[f(\alpha(t)) - f(\widehat{\alpha}(t))\right]\right| \\ &\leq 2M_f \mathbb{P}_{x,i,i}\{\vartheta \wedge \tau_h \leq t\} \leq 2M_f\varepsilon, \text{ for } t \in [0,T]. \end{aligned}$$

where $M_f = \sup_{i\in\mathbb{Z}_+} |f(i)|$. The lemma is proved. □

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# ABSTRACT

## SWITCHING DIFFUSION SYSTEMS WITH PAST-DEPENDENT SWITCHING HAVING A COUNTABLE STATE SPACE

by

**HAI DANG NGUYEN**

**August 2018**

**Advisor:** Dr. G. George Yin

**Major:** Mathematics (Applied)

**Degree:** Doctor of Philosophy

Emerging and existing applications in wireless communications, queueing networks, biological models, financial engineering, and social networks demand the mathematical modeling and analysis of hybrid models in which continuous dynamics and discrete events coexist. Assuming that the systems are in continuous times, stemming from stochastic-differential-equation-based models and random discrete events, switching diffusions come into being. In such systems, continuous states and discrete events (discrete states) coexist and interact.

A switching diffusion is a two-component process $(X(t), \alpha(t))$, a continuous component and a discrete component taking values in a discrete set (a set consisting of isolated points). When the discrete component takes a value $i$ (i.e., $\alpha(t) = i$), the continuous component $X(t)$ evolves according to the diffusion process whose drift and diffusion coefficients depend on $i$. Until very recently, in most of the literature $\alpha(t)$ was assumed to be a process taking values in a finite set, and that the switching rates of $\alpha(t)$ are either independent or depend only on

the current state of $X(t)$. To be able to treat more realistic models and to broaden the applicability, this dissertation undertakes the task of investigating the dynamics of $(X(t), \alpha(t))$ in a much more general setting in which $\alpha(t)$ has a countable state space and its switching intensities depend on the history of the continuous component $X(t)$. We systematically established important properties of this system: well-posedness, the Markov Feller property, and the recurrence and ergodicity of the associated function-valued process. We have also studied several types of stability for the system.

# AUTOBIOGRAPHICAL STATEMENT

Hai Dang Nguyen

## Education

- *Ph.D. in Applied Mathematics*, Wayne State University, May 2018 (expected)
- *M.A. in Mathematical Statistics*, Wayne State University (2017)
- *M.S. in Probability and Statistics*, Vietnam National University, Hanoi (2011)
- *B.A. in Mathematics* , Vietnam National University, Hanoi, Vietnam (2008)

## Awards

- SIAM Travel Award, 2017.
- The Karl W. & Helen L. Folley Endowed Scholarship, Wayne State University, 2017.
- Thomas C. Rumble University Graduate Fellowship, Wayne State University, 2016-2017.
- The Maurice J. Zelonka Endowed Scholarship, Wayne State University, 2015.
- 2nd Prize in Student Research Award, VNU, Hanoi University of Science, 2008.
- Odon Vallet Scholarship, Vietnam, 2007.
- 2nd Prize for Analysis at the National Mathematical Students Olympiad, Vietnam, 2005.

## Dissertation-Related Publications

1. D. H. Nguyen, G. Yin, Modelling and analysis of switching diffusion systems: Past-dependent switching having a countable state space, *SIAM J. Control Optim.* 54 (2016), no. 5, 2450-2477.
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